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Technical Section

SHOCK PRESERVING QUADRATIC INTERPOLATION FOR VISUALIZATION ON TRIANGULAR MESHES

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Abstract—A new interpolation method for visualizing shock-type solutions on triangular meshes is presented. The method combines standard linear and quadratic interpolants in such a way as to avoid spurious numerical values. The effectiveness of the method is demonstrated on test problems. Copyright © 1996 Elsevier Science Ltd

1. INTRODUCTION

Many areas of scientific computing involve modelling real world problems. The visualization of the solution to these problems is an essential aid in the understanding of the phenomenon being modelled. Interpolation schemes that will respect the physical properties of the underlying data are thus needed. One example of respecting this physical nature of the data is to produce values within a specific range, for example to ensure positivity.

Many problems that require such treatment can be modelled by differential equations, either ordinary differential equations (ODEs) or partial differential equations (PDEs). An important feature of these problems is that initial smooth conditions may develop into shocks and discontinuities. Some interpolation schemes may produce results that introduce physically unreal values for such problems.

A wide range of numerical software exists for solving such problems in one spatial dimension, for example the NAG numerical library provides routines to solve ODEs. A number of papers have therefore discussed the preservation of inherent properties of data arising from the solution of ODEs, see Brankin and Gladwell [1] for preservation of convexity, Higham [2] for monotonicity and Butt and Brodlie [3] for preservation of positivity.

In two spatial dimensions a number of general purpose PDE solvers are becoming available, such schemes use triangular elements because they can accurately represent complex domains and may be used in conjunction with adaptive spatial meshes. A number of authors use a cell-centered finite volume spatial discretization scheme to solve the large class of convection-dominated PDEs, see [4–7] for details. Other numerical schemes such as the finite element method may also be used to solve such problems using triangular elements.

The numerical solution of convection-dominated problems requires great care in the preservation of the shape of the solution. The avoidance of spurious oscillations around shocks is of great importance. Often schemes will reduce the order of accuracy around these features to preserve the extrema. The reduction of linear to piecewise constant, limiting higher order terms of polynomials [8], or limiting solution values themselves, are all used to ensure that no new extrema are created. Interpolation schemes used in conjunction with such problems must take great care not to introduce these unwanted features.

Interpolation performed in a standard way will not provide this desired property and cause overshoots at shocks. The importance of eliminating spurious oscillations is such that accuracy is often forfeit to ensure this. A modified quadratic scheme is proposed that is prepared to forgo interpolation of the midpoint values to control this behaviour.

This paper will describe a triangular based quadratic interpolant that is bounded by the minimum and maximum value defining it. The approximation of the surface this produces will preserve the inherent shape of the data and guarantee that all values lie within a given range. The new interpolant may therefore be utilized to preserve positivity. It can also be used for the visualization of the solution and by the numerical solver to recover values over the numerical domain.

2. AN EXAMPLE PROBLEM

Consider the following PDE defined by

$$\frac{\partial u}{\partial t} + w(x,t)\frac{\partial u}{\partial x} + w(y,t)\frac{\partial u}{\partial y} - \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0,$$
$$\nu = 0.0001$$

The solution domain is $[0,1]\times[0,1]$ in space and $(0,1)$ in time. This is the Burgers' equation, see [9], having

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the exact solution given by

$$u(x,y,t) = w(x,t)w(y,t) \text{ where } w(x,t) = \frac{0.1A + 0.5B + C}{A + B + C}$$

and

$$A = e^{-0.005x-0.5+4.95t)/\nu},$$
$$B = e^{-0.25(x-0.5+0.75t)/\nu)},$$
$$C = e^{-0.5(x-0.375)/\nu}$$

The function represents two wave fronts moving across the numerical domain. Consider a regular mesh of 1458 triangles over this domain. Data values are given at each node of the mesh and also at the edge midpoints of each triangle. The data values given at these points define the physical range of the problem. The problem is taken at time $t=0.45$. Figure 1 shows a surface plot of the function. All surface plots are produced by the Unimap package using a 81×81 regular grid as input. The plot shows the information without performing any additional interpolation.

Standard quadratic interpolation techniques [10] can be used to fit six shape functions over each triangle to give an approximation to the surface. A 2-D quadratic interpolant needs six data points: these points are usually at the vertices of the triangle and the mid-points of the sides. These can be mapped to area coordinates (L_1, L_2, L_3). Six shape functions can be fitted to these points such that they are unity at one point and vanish at the others. These shape functions are shown in Table 1 and have the property that they sum to unity. The interpolated value f is then defined by

$$f = \sum_{i=1}^{6} \phi_i f_i$$

However, given that there is a physical range to the problem, where does the standard interpolation method fail to observe this? If the physical range of the data is defined by the interpolation data used then Fig. 2 shows the areas where this interpolation scheme produces values outside this range. Fig. 3 shows how the proposed modified quadratic interpolant observes this range and produces no values outside the interpolation data range.

3. SHOCK PRESERVING INTERPOLATION

Two other important interpolation methods in this area are those of Abgrall [11] and Barth [7]. Both these schemes have the common approach of using adaptive multi-triangle stencils to achieve high

Table 1. The standard (left) [9] and modified (right) quadratic shape functions

$\phi_1 = (2L_1 - 1)L_1$	$\hat{\phi}_1 = L_1^2$
$\phi_2 = (2L_2 - 1)L_2$	$\hat{\phi}_2 = L_2^2$
$\phi_3 = (2L_3 - 1)L_3$	$\hat{\phi}_3 = L_3^2$
$\phi_4 = 4L_2L_3$	$\hat{\phi}_4 = 2L_2L_3$
$\phi_5 = 4L_3L_1$	$\hat{\phi}_5 = 2L_3L_1$
$\phi_6 = 4L_1L_2$	$\hat{\phi}_6 = 2L_1L_2$

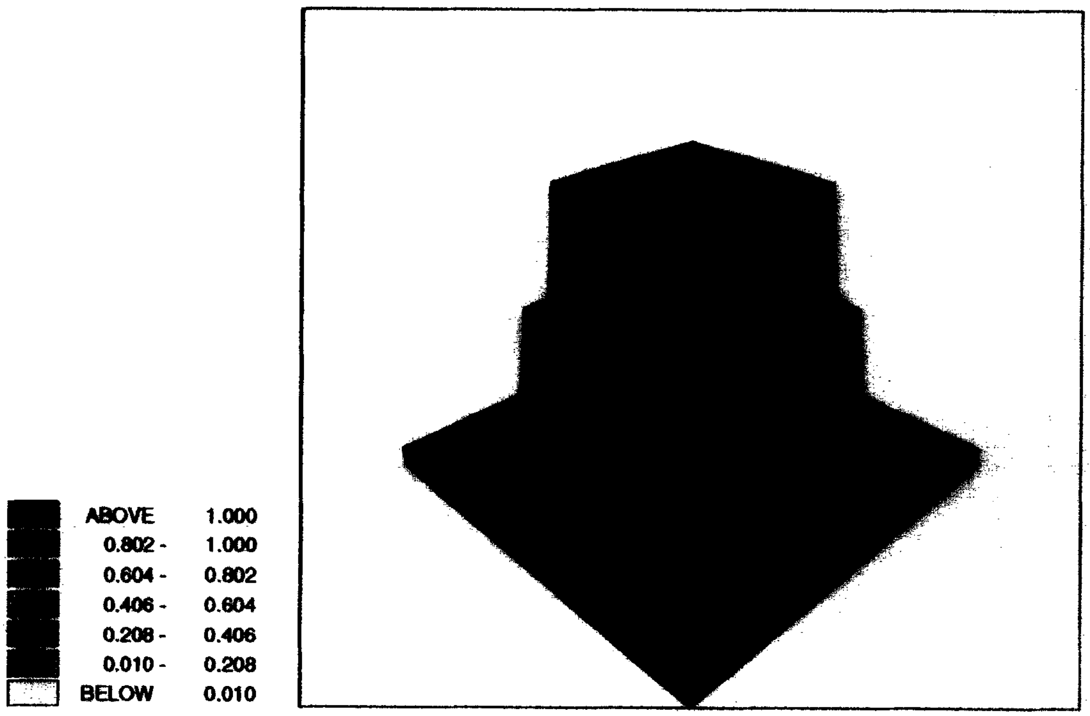

Fig. 1. The true Burgers' equation.

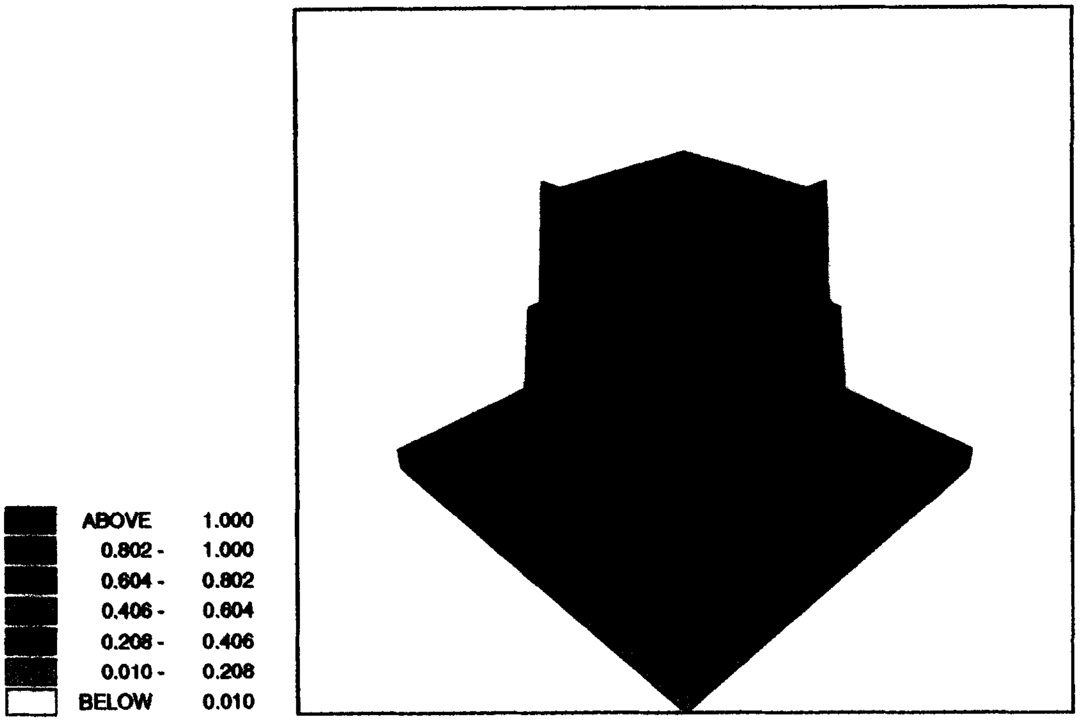

Fig. 2. Standard quadratic interpolant Burgers' equation.

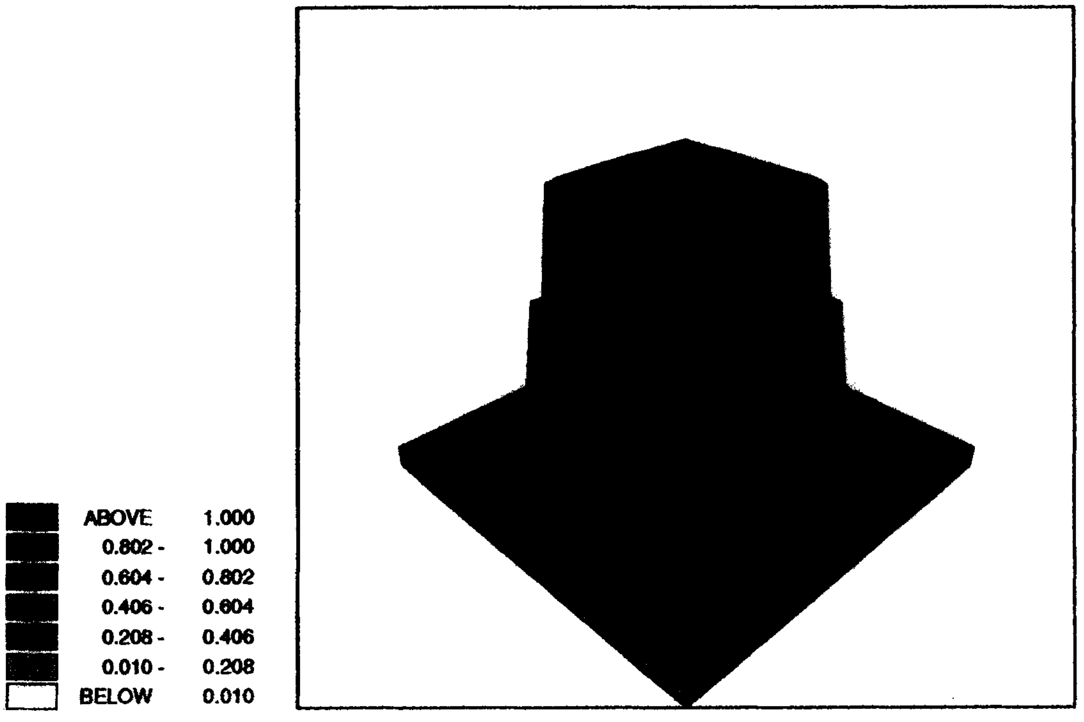

Fig. 3. Modified quadratic interpolant Burgers' equation.

order accuracy for problems which may have shocks and discontinuities. Abgrall's adaptive essentially non-oscillatory (ENO) scheme takes the form of either the centroid of a triangle acting as a control volume for that triangle or the construction of control volumes around each node in the mesh. The method involves the construction of an interpolant of order n by several steps. The initial step involves the construction of a linear interpolant and each following step will increase the order of the interpolant. Several possible combinations of points can be used at each step. The choice of which points to use is made by examining the coefficients of the Lagrange polynomials constructed from each combination. The set chosen is the one in which the sum of the absolute values of the coefficients of the Lagrange polynomial is minimal. The problem is that the number of possible combinations grows rapidly, and the stencil used is potentially large. Abgrall controls the choice of the additional values considered at each step by only considering neighbouring nodes of nodes already chosen. Even so the growth is rapid and the points considered may be far removed from the original node. Abgrall's good results provide a more than adequate justification of the scheme however.

The aim here is to consider a simpler alternative to Abgrall's scheme. A quadratic interpolant is constructed which avoids introducing new extrema by modifying the standard quadratic shape functions over each triangle. The result of this modification is that the new interpolant may not pass through all the data points used to define it. This use of the interpolation information as control points rather than data points is not uncommon in other forms of interpolation. Bézier curves must lie within the convex hull of the corresponding Bézier polynomial [12]. The justification for this is that other properties of the curve are more important, in this case the curve is aesthetically pleasing. In fact, the Bézier curve is constructed from Bernstein basis functions which are all positive and sum to unity over the parametric coordinates. The approach taken here thus has some similarities with Bézier interpolants.

3.1. *The standard and modified quadratic interpolation schemes*

The problem with the standard interpolation is that the shape functions associated with the three vertex values are negative over large parts of the triangle. Thus it is possible for new extrema to be introduced. This is unsatisfactory for shock problems (Fig. 2 clearly shows this).

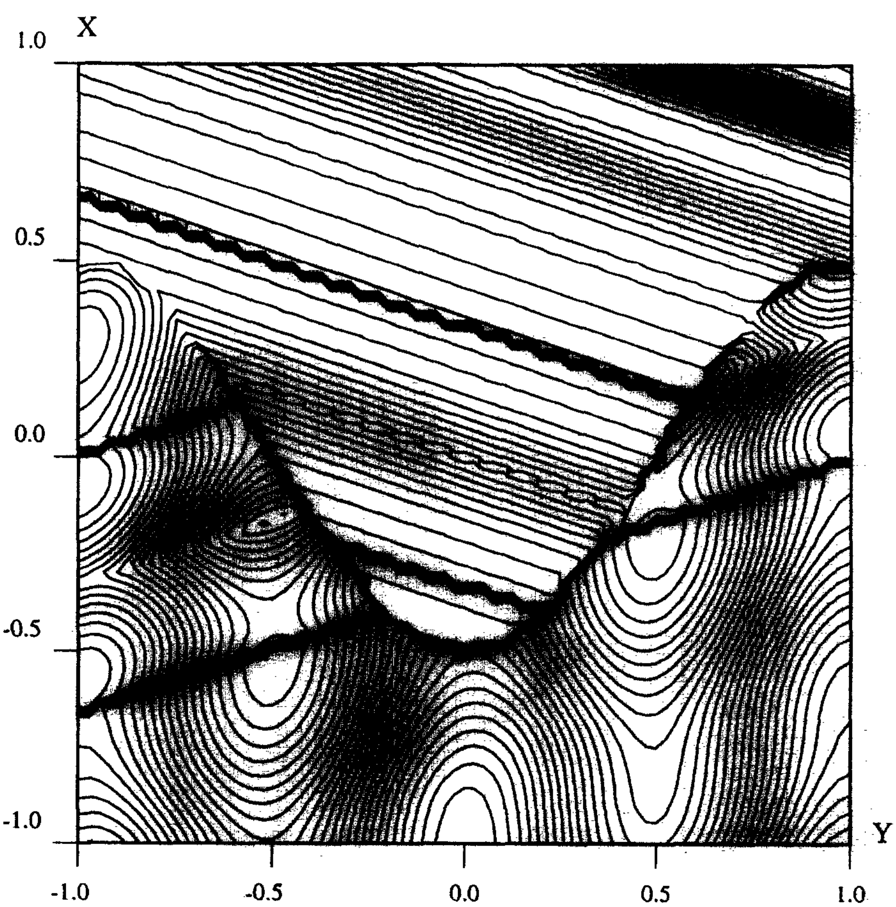

Fig. 4. Contour plot of the true function.

The elimination of the possibility of creating new extrema is achieved by ensuring that all shape functions are bounded by the constraint, $0 \leqslant \phi_i \leqslant 1$ and also by maintaining the condition that $\Sigma\phi_i = 1$ for $i = 1, \ldots, 6$.

The value produced by a linear interpolant, f_L, will satisfy these constraints. The three linear shape functions are always positive and sum to unity over the triangle. However, the value given by the standard quadratic interpolant, f_s, does not satisfy the required constraints and can therefore create new extrema and can cause overshoot near shocks.

Given the modified interpolant can be written as

$$f_M = \alpha f_L + (1 - \alpha) f_S$$

then a value of $\alpha = 1$ will give the standard linear interpolant and a value $\alpha = 0$ will give the standard quadratic. The question is what range of values will always ensure the modified scheme displays the desired behaviour? To guarantee that the quadratic shape functions remain positive over each triangle a value of α between the range $1/2 \leqslant \alpha \leqslant 1$ must be chosen. Under the assumption that a quadratic interpolant will produce better results than a linear interpolant, then the smallest value of α is taken. The resulting six shape functions are shown in Table 1. These shape functions have the properties that $0 \leqslant \hat{\phi}_i \leqslant 1$ and $\sum \hat{\phi}_i = 1$ for $i = 1, \ldots, 6$. All visual and numerical results shown here use the value of $\alpha = 1/2$. Selecting $\alpha = 1/2$, the new modified shape functions $\hat{\phi}_1, \hat{\phi}_2$ and $\hat{\phi}_3$ no longer vanish at the adjacent edge mid-points but have a value of 1/4. The mid-point shape functions do vanish at all other points but are not unity at their associated point (the actual value is 1/2).However, the solution methods used to solve these problems take great care to respect the physical properties of the solution. The elimination of spurious oscillations is a prime concern. The modified interpolant will ensure that no new extrema are created. It is this property that is of special interest—the ability of the scheme to not interpolate solution values outside the range of existing values.

The use of $\alpha = 1/2$ will always guarantee that the interpolant will be bounded. This is also true when considering the one dimensional quadratic problem. An area for future work is to select a larger value of α which, depends upon the six data values given to the interpolant, and which will still satisfy the no-new-extrema constraint.

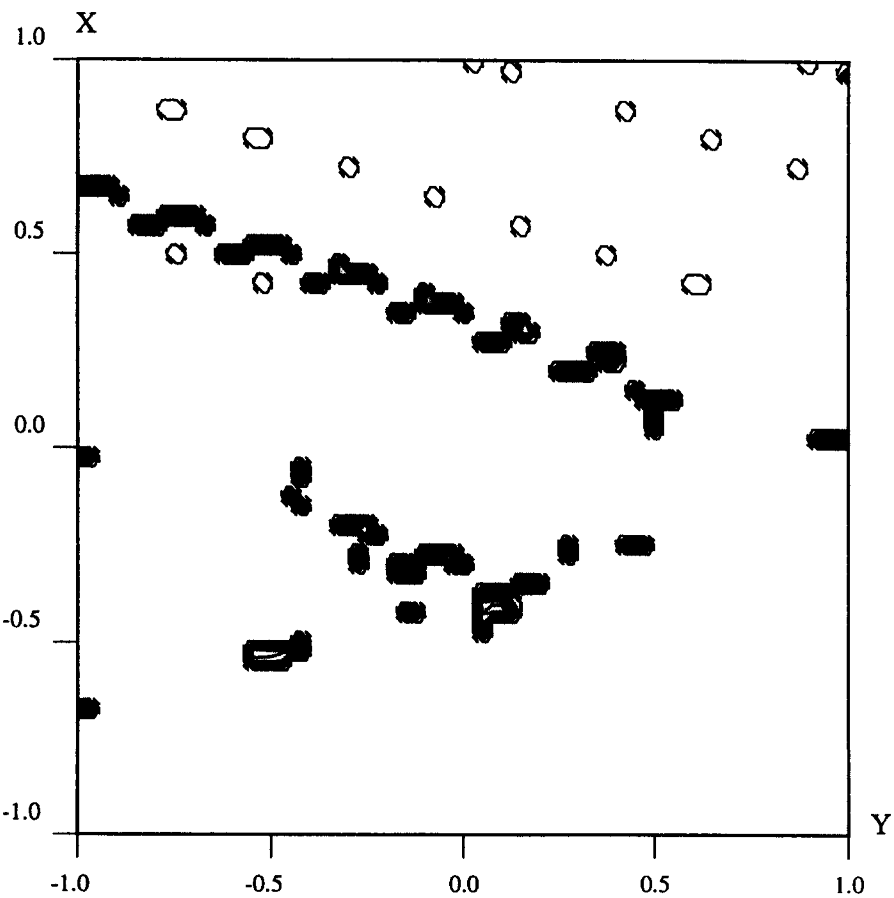

Fig. 5. Contour plot of the overshoot and undershoot when using standard quadratic interpolation.

4. ERROR ANALYSIS

The modified shape functions give an error at the mid-point values. If the 1-D situation along the edge of the triangle is considered then the modified interpolant along the edge can be defined, in parametric coordinates, as

$$f_1(1-L)^2 + f_2 2L(1-L) + f_3 L^2$$

At the mid-point, $L = 1/2$, the interpolated value, f^*, is given by

$$f^* \begin{array}{l} = \frac{1}{4}f_1 + \frac{1}{2}f_2 + \frac{1}{4}f_3 \\ = f_2 + \frac{1}{4}(f_1 - 2f_2 + f_3) \\ \approx f_2 + \frac{h^2}{4}f_2'' + O(h^4) \end{array}$$

where f_1 and f_3 are the vertex values, f_2 is the edge mid-point value, h is the distance between data points and assuming that the function defined by f_1, f_2, f_3 has a second derivative. An error is introduced which is proportional to the distance between data points and the second derivative of the function at f_2, the mid-point. Although this approach gives an error of the same order as the linear interpolant the difference in accuracy is significant, the modified quadratic interpolant producing superior results on test problems, this will be shown in Section 5.

Table 2.

Number of triangles	162	1458	13122
Standard quadratic	1.3038e-02	6.0654e-03	1.9130e-03
Modified quadratic	1.2201e-02	8.2464e-03	2.3021e-03
Linear interpolant	1.4525e-02	1.3601e-02	3.6575e-03

5. RESULTS

The modified interpolant was compared to the standard interpolant on a number of examples. Numerical results were obtained to demonstrate that the expected order of convergence is not observed when dealing with problems that contain shocks and discontinuities. All numerical results were obtained over a [0, 1] × [0, 1] domain except for the complex function in Section 5.3 which has a [−1, 1] × [−1, 1] domain. A regular mesh with $2N^2$ triangles is fitted over the domain. A L_1 norm is used with a seven point fifth order Gaussian quadrature scheme [13] for the numerical integration over each triangle to measure the error over the domain.

5.1. *Burgers' equation*

The results for the Burgers' equation defined in Section 2 are given in Table 2. This illustrates how

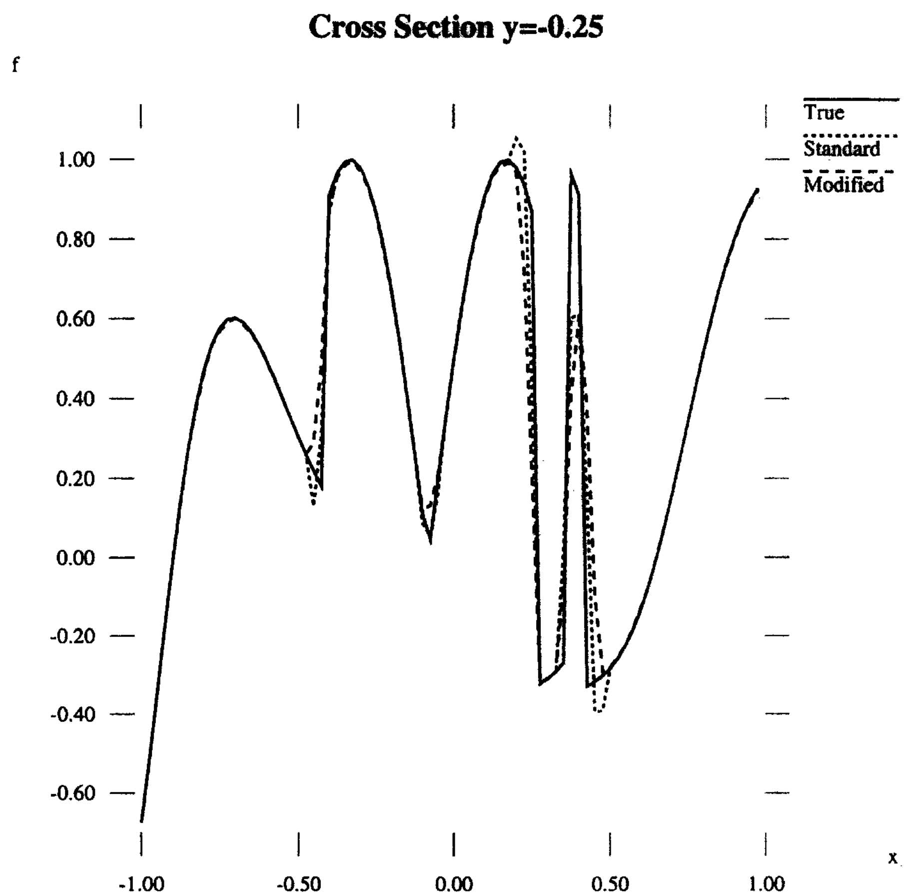

Fig. 6. Cross section using a triangular mesh with 1458 elements.

the expected order of convergence is not observed when looking at a problem with shock waves and discontinuities.

5.2. *Anisotropy problem*

The second function is defined by

$$u(x,y,t) = \frac{3}{4} - \frac{1}{4+4e^{B}}$$

where $B = 0.125(-x + y - 0.75t)/v$

and is the exact solution to a PDE similar to that used by Zegeling [14],

$$\frac{\partial u}{\partial t} + 3u\frac{\partial u}{\partial x} + 3(1.5-u)\frac{\partial u}{\partial y} - 3v\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$= 0, v = 1.0 \times 10^{-9}$$

The solution domain is [0, 1]×[0, 1] in space and (0, 1) in time. The solution is taken at time t=0.5, results are shown in Table 3.

This again illustrates how the expected order of convergence is not observed when looking at problems with shock waves and discontinuities.

5.3. *Complex function*

Consider the following function from [11].

Table 3.

Number of triangles	162	1458	13122
Standard quadratic	3.6819e-03	8.1661e-04	4.0437e-04
Modified quadratic	3.5060e-03	1.1664e-03	3.8684e-04
Linear interpolant	4.5331e-03	1.9324e-03	5.1262e-04

If Θ is any angle, let f_Θ be

$$f_\theta = \begin{cases} if\, r & \leqslant -1/3 \\ & f_\theta(x,y) = -r\sin(\frac{3\pi r^2}{2}) \\ if\, r & \geqslant 1/3 \\ & f_\theta(x,y) = 2r - 1 + \frac{\sin(3\pi r)}{6} \\ if\, |r| & < 1/3 \\ & f_\theta(x,y) = |\sin(2\pi r)| \end{cases}$$

where $r = x - \frac{\cos(\theta)}{\sin(\theta)}y$ and let $u(x,y)$ be:

$$(x,y) = \begin{cases} if\, x & \leqslant 1/2\cos(\pi y) \\ & u(x,y) = f_{\sqrt{\pi/2}}(x,y) \\ if\, x & > 1/2\cos(\pi y) \\ & u(x,y) = f_{-\sqrt{\pi/2}}(x,y) + \cos(2\pi y) \end{cases}$$

This function has many discontinuities and provides a very challenging example for interpolation schemes.

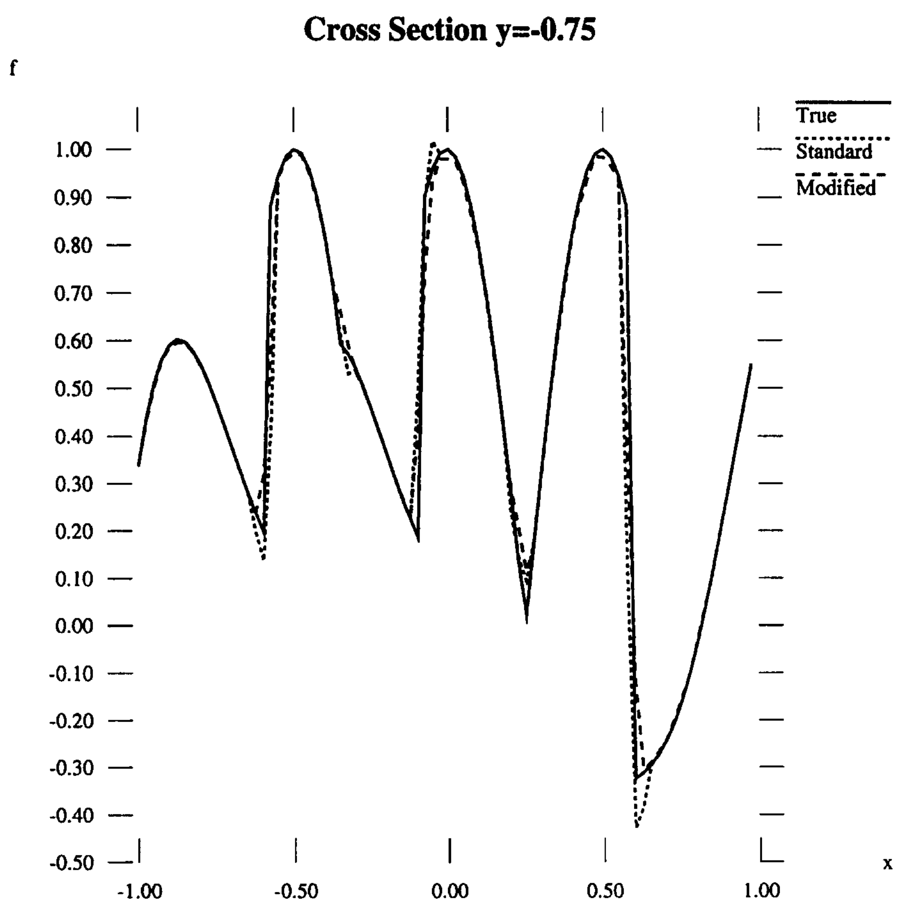

Fig. 7. Cross section using a triangular mesh with 1458 elements.

A contour plot of the solution is shown in Fig. 4 while Fig. 5 shows where the undershoots and overshoots are found when traditional interpolation is used, *i.e.* when the value produced by the standard interpolant is outside the local range of the true function. The areas of overshoot can be seen to coincide with the discontinuities present in the function. Both plots were produced by Xprism3, part of the Khoros visualization software suite. The plots use 32 contour lines and were produced from a regular grid of 81×81 over the domain.

Figures 6 and 7 show two cross sections through the function at $y = -0.25$ and at $y = -0.75$. These show the magnitude of the overshoot when using the standard interpolant and the way the modified interpolant respects the physical nature of the problem. The true function, the standard interpolant and the modified interpolant are shown.

6. CONCLUSIONS

This new method, based on modified quadratic shape functions, will not create any new extrema in the data since they are always positive and sum to unity. In the applications of interest, the new method also has the same rate of convergence on difficult problems, *i.e.* those with steep gradients as an unmodified quadratic. The modified quadratic also has the important advantage that the interpolant created will be bounded by the maximum and minimum function values used to define it. This factor has been a key requirement when considering the problem class the interpolant is designed for. However, ensuring these properties results in the greater dependence of the interpolant upon the vertex based shape functions than the mid-point shape functions.

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