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A Closed-Form Solution to Single Scattering for General Phase Functions and Light Distributions

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Abstract

Due to the intricate nature of the equation governing light transport in participating media, accurately and efficiently simulating radiative energy transfer remains very challenging in spite of its broad range of applications. As an alternative to traditional numerical estimation methods such as ray-marching and volume-slicing, a few analytical approaches to solving single scattering have been proposed but current techniques are limited to the assumption of isotropy, rely on simplifying approximations and/or require substantial numerical precomputation and storage. In this paper, we present the very first closed-form solution to the air-light integral in homogeneous media for general 1-D anisotropic phase functions and punctual light sources. By addressing an open problem in the overall light transport literature, this novel theoretical result enables the analytical computation of exact solutions to complex scattering phenomena while achieving semi-interactive performance on graphics hardware for several common scattering modes.

Categories and Subject Descriptors (according to ACM CCS): I.3.7 [Computer Graphics]: Three-Dimensional Graphics and Realism—Color, shading, shadowing, and texture

1. Introduction

As they mathematically model radiative transfer in a vast scope of natural elements, simulating light transport in participating media has considerable scientific implications. Numerous applications exist in the entertainment industry where visually appealing and realistic rendering of underwater scenes, god rays and other atmospheric phenomena drive the next generation of special effects in movies, animated films and video-games. Reliable simulations are also crucial in the automotive industry and architectural design when conceiving car headlamps or stage lighting for instance, as well as in safety-oriented research in order to assess the effectiveness of emergency exit signs in a smoke-filled room or the visibility of traffic signs by foggy weather.

Unfortunately, the intricacy of the radiative transport equation results in a formulation notoriously complex to evaluate, making the efficient simulation of accurate light transport in participating media very challenging. While numerical methods such as volume-slicing and ray-marching provide a generic means of converging towards the solutions to radiative transfer problems, analytical approaches to solving single scattering have recently received attention in the graphics community and were shown to be a very promising alternative to the former traditional rendering techniques. However, current (semi-)analytical methods are limited to the assumption of isotropy, rely on simplifying approximations and/or require substantial numerical precomputation and storage of the sampled results.

In this paper, we present the first closed-form solution to the air-light integral in homogeneous media for general azimuthally symmetrical anisotropic phase functions and punctual light sources. Assuming a generic representation of angular distributions, we provide a mathematical formulation of the problem and show how the latter may be effectively reduced to the knowledge of a single indefinite integral. In order to determine the antiderivative of interest, we then introduce a novel analytical derivation leading to the formulation of a closed-form solution which, to the best of our knowledge, has never appeared before in the literature.

This new theoretical result permits the analytical simulation of complex scattering phenomena which, up until now, still represented an open problem in the overall light trans-

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port literature. Unlike the many approximation methods proposed over the last 30 years, the technique enables the exact calculation of ground-truth solutions in a finite number of computational steps while still achieving semi-interactive performance on current-generation graphics hardware for several common scattering modes.

After reviewing previous work relating to the topic of concern, this document provides an overview of the theoretical background on the air-light integral. The assumed generic representation of angular distributions is then introduced followed by a mathematical formulation of the problem and the derivation of our closed-form solution. Finally, results evaluating both the accuracy and the performance characteristics of the method are presented before discussing future work orientations.

2. Related Work

Participating media rendering has led to a wide variety of techniques within the computer graphics literature, and those most related to our single scattering model are summarized below while referring the interested reader to [CPCP*05] for a more complete survey. Following the work of Max [Max86a], several methods based on volume-slicing [DYN00, DYN02, IJTN07] or ray-marching [NMN87, Mak08, ED10] have been developed. While these numerical techniques are both general and simple, such estimation schemes are however inherently prone to undersampling artifacts due to the Riemann summation that they rely on. On the other hand, the pioneering work of Blinn [Bli82] was the precursor to several analytical methods under the assumption of a directional light source [Max86b, Wil87, HP02, HP03, REK*04], while an image-based post-processing technique was alternatively presented [Mit07].

Such an analytical approach was later proposed for point light sources in homogeneous media by Lecocq et al. [LMAK00] who reformulated the air-light integral under an angular parameterization. The integration then relies on a Taylor approximation of the integrand yielding a semi-analytical solution for isotropic light sources while anisotropic distributions require numerical precomputation. Subsequent extensions [BAM06] also took visibility/occlusion of the light source into account by limiting the integration domain to illuminated segments along a ray using shadow-volumes.

Later, Sun et al. [SRNN05] simplified the angular formulation further and proposed to tabulate sampled solutions of the integral. Although real-time performance may be achieved on graphics hardware, this simple and practical approach nevertheless entails several limitations. First, the method relies on lengthy numerical precomputation and storage of the solution for various discrete sample points in the continuous 2-D domain. During rendering, the result is then interpolated to recover missing data between the samples which leads to visible artifacts. Additionally, the data may only be tabulated for a finite subset of the semi-infinite 2-D space, and it is therefore impossible to guarantee that the range of parameters needed for an arbitrary application will be covered. The technique is then constrained to resort to extrapolation which yields further inaccuracies. Finally, the method is limited to isotropic light sources, and its extension to anisotropic phase functions requires the precomputation and storage of one additional table for each degree of the phase function representation, therefore increasing precomputation cost and putting additional pressure on memory requirements. The approach was then hybridized with ray-marching to render light-shafts [WR08] while Zhou et al. [ZHG*07] alternatively proposed an analytical approximation for inhomogeneous media but under the restriction of isotropy.

More recently, Pegoraro and Parker [PP09] presented a closed-form solution to the air-light integral for isotropic phase functions and light sources. The approach was subsequently extended to 1-D anisotropic functions by deriving a dual-formulation of the air-light integral [PSP09a] which was further optimized in [PSP09b]. The semi-analytical solution however relies on a Taylor series expansion of angular distributions in the integration space which induces several limitations. First, the formulation of the recursive derivatives requires a human intervention, and while this is doable for the first few terms of simple functions such as a spotlight, the process becomes truly prohibitive for complex anisotropic distributions due to the rapidly increasing length of the resulting expressions. Moreover, since physical phase functions or light distributions can generally not be faithfully represented by a polynomial in the integration domain, the approximation is inherently prone to inaccuracies.

Although numerous methods have been proposed over the last 3 decades, the problem of concern has until now solely been approached by numerical or (semi-)analytical approximations all exhibiting various inherent limitations, hence justifying the ultimate need for a theoretically correct computational model. Based on the concepts introduced by Pegoraro et al., this paper presents the very first closed-form solution to single scattering for general rotationally symmetrical angular distributions.

3. Closed-Form Single Scattering

This section first provides an overview of the theoretical background on the air-light integral in homogeneous participating media considering arbitrary phase functions and light sources. The assumed generic representation of their angular distributions is subsequently introduced before presenting a mathematical formulation of the problem and the derivation of our closed-form solution. As in previous analytical approaches [LMAK00, SRNN05, PSP09a], the analysis to follow solely treats of fully-illuminated ray segments while referring the reader to the discussion section for an explanation on how this restriction may be easily lifted.

3.1. Air-Light Integral

The air-light integral describes the radiance L at position x_a along a ray of direction $\vec{\omega}$ through a homogeneous medium with background boundary at x_b as

$$L(x_a, \vec{\omega}) = L_m(x_a, x_b, \vec{\omega}) + L_r(x_a, x_b, \vec{\omega}). \quad (1)$$

Defining the extinction coefficient $\kappa_t = \kappa_a + \kappa_s$ with κ_a and κ_s being the absorption and scattering coefficient respectively, the reduced radiance reads

$$L_r(x_a, x_b, \vec{\omega}) = e^{-\kappa_t (x_b - x_a)} L(x_b, \vec{\omega}). \quad (2)$$

If the medium has a phase function Φ and is solely illuminated by a punctual light source of intensity I located at position $\vec{p}_l$ and parameterized by the angle with the normalized direction $\vec{v}_l$ as recapitulated in table 1, then the formulation of the medium radiance along the view ray of origin $\vec{p}_e$ and direction $\vec{v}_e$ reads [NMN87, PSP09a]

$$L_m(x_a, x_b, \vec{\omega}) = \kappa_s e^{\kappa_t x_a} \int_{x_a}^{x_b} \frac{e^{-\kappa_t \left(x + \sqrt{h^2 + (x - x_h)^2}\right)}}{h^2 + (x - x_h)^2} \quad (3)$$
$$\Phi\left(\arctan\left(\frac{x - x_h}{h}\right) + \frac{\pi}{2}\right) I \left(\arccos\left(\frac{d_{el} x + d_{lel}}{\sqrt{h^2 + (x - x_h)^2}}\right)\right) \mathrm{d}x$$

where $d_{el} = \vec{v}_e \cdot \vec{v}_l$ and $d_{lel} = (\vec{p}_e - \vec{p}_l) \cdot \vec{v}_l$, and where h is the distance from the light to the ray and x_h the coordinate of its projection onto it as illustrated in figure 1.

The medium radiance may then be simplified by substituting $u = \frac{x - x_h}{h}$ such that the integrand becomes a function of only 4 parameters [PSP09a]

$$L_m(x_a, x_b, \vec{\omega}) = \frac{\kappa_s}{h} e^{\kappa_t (x_a - x_h)} \int_{u_a}^{u_b} \frac{e^{-H\left(u + \sqrt{1 + u^2}\right)}}{1 + u^2} \quad (4)$$
$$\Phi\left(\arctan(u) + \frac{\pi}{2}\right) I \left(\arccos\left(\frac{d_{el} u + d_c}{\sqrt{1 + u^2}}\right)\right) \mathrm{d}u$$

where $H = \kappa_t h$ is the optical distance from the light to the ray, $d_c = \frac{d_{el} x_h + d_{lel}}{h} = \vec{v}_h \cdot \vec{v}_l$ with $\vec{v}_h = \frac{(\vec{p}_e - \vec{p}_l) + x_h \vec{v}_e}{h}$ being the unit-length projection vector of the light onto the ray, and

$$u_a = \frac{x_a - x_h}{h} \qquad u_b = \frac{x_b - x_h}{h}. \quad (5)$$

Defining Φ_c such that $\Phi(\theta) = \Phi_c(\cos(\theta))$ and I_c such that $I(\vartheta) = I_c(\cos(\vartheta))$, the substitution $v = u + \sqrt{1 + u^2}$ then yields [PSP09a]

$$L_m(x_a, x_b, \vec{\omega}) = \frac{\kappa_s}{h} e^{\kappa_t (x_a - x_h)} 2 \int_{v_a}^{v_b} \frac{e^{-Hv}}{v^2 + 1} \quad (6)$$
$$\Phi_c\left(-\frac{v^2 - 1}{v^2 + 1}\right) I_c\left(\frac{d_{el}(v^2 - 1) + 2 d_c v}{v^2 + 1}\right) \mathrm{d}v$$

where

$$v_a = u_a + \sqrt{1 + u_a^2} \qquad v_b = u_b + \sqrt{1 + u_b^2}. \quad (7)$$

Since the focus of this work is the evaluation of the medium radiance, extinction phenomena are assumed to prevail in surface shading calculation. Defining β as the bidirec-

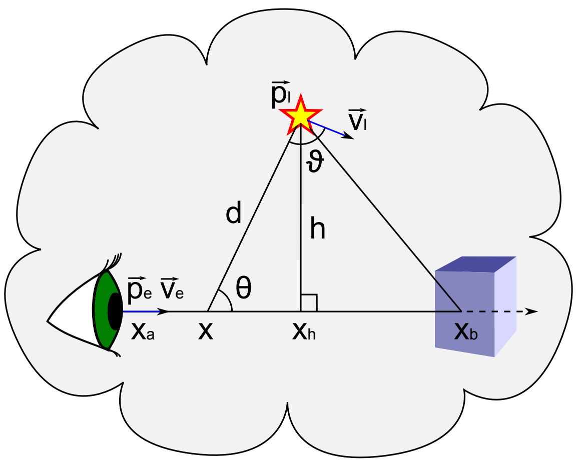

Figure 1: *Diagram illustrating the various quantities that the air-light integral depends on.*

Table 1: *Summary of the notation used in this document.*

Symbol	Description
L_m	Medium radiance
L_r	Reduced radiance
$\vec{p}_e$	Eye position
$\vec{v}_e$	Eye ray direction
$\vec{p}_l$	Light position
$\vec{v}_l$	Light ray direction
$\vec{v}_h$	Projection vector of $\vec{p}_l$ onto the eye ray
x_a	Lower bound of the integration domain
x_b	Upper bound of the integration domain
x_h	Projection coordinate of $\vec{p}_l$ onto the eye ray
h	Distance from $\vec{p}_l$ to the eye ray
κ_a	Absorption coefficient
κ_s	Scattering coefficient
κ_t	Extinction coefficient
H	Optical distance from $\vec{p}_l$ to the eye ray
d_{el}	Cosine angle between $\vec{v}_e$ and $\vec{v}_l$
d_{lel}	Projection coordinate of $\vec{p}_e$ onto the light ray
d_c	Cosine angle between $\vec{v}_h$ and $\vec{v}_l$
Φ	Phase function
I	Light intensity distribution
θ	Phase function parameter angle
ϑ	Light distribution parameter angle
N_Φ	Number of phase function coefficients
c_Φ	Phase function coefficients
N_I	Number of light source coefficients
c_I	Light source coefficients

tional reflectance distribution function, the background radiance then reads

$$L(x_b,\vec{\omega}) = \frac{e^{-\kappa_t d(x_b,\vec{\omega})}}{d(x_b,\vec{\omega})^2}\,\beta(\vec{\omega},\vec{\omega}_i,\vec{n}_b)\,\vec{\omega}_i\cdot\vec{n}_b\,I_c(-\vec{\omega}_i\cdot\vec{v}_l) \quad (8)$$

where $d(x,\vec{\omega}) = \sqrt{h^2+(x-x_h)^2}$ represents the distance from the light source to a point x along the ray, $\vec{\omega}_i$ is a unit-length vector directed towards the light source, and $\vec{n}_b$ is the surface normal at x_b.

Considering the above problem formulation from Pegoraro et al. [PSP09a], the remainder of this document focuses on addressing the evaluation of the medium radiance, and this is precisely where our approach fundamentally differs from theirs. The latter relies on an expansion of the phase function and light distribution in the *integration space*, i.e. on a representation as a polynomial of the variable v such that $\Phi_c()I_c() = \sum_{n=0}^{N-1} c_n v^n$. While the assumption simplifies the antiderivatives to be solved, actual angular distributions are however typically not exactly expressible in such form, therefore involving inherent approximations. On the other hand, we consider a polynomial basis in the *cosine space* as formulated in equations 9 and 10, which is the native representation of most phase functions and many light distributions as discussed in section 3.2. Doing so results in rational expressions in the integration space as illustrated in equation 20 which contrasts with the aforementioned polynomial of the variable v assumed by Pegoraro et al. [PSP09a]. We then analyze the whole formulation of the problem in section 3.3 and provide a significantly more general mathematical solution to the resulting antiderivative enabling the air-light integral to be solved in closed form as detailed in section 3.4.

3.2. Representing Angular Distributions

In order to accurately model a wide range of arbitrary phase functions and light sources, we assume that the angular distributions are expressible via a generic formulation. A broadly used representation describes the phase function as an N_Φ-term polynomial of the cosine angle $\mu = \cos(\theta)$ as follows

$$\Phi_c(\mu) = \sum_{n=0}^{N_\Phi-1} c_\Phi(n)\mu^n. \quad (9)$$

Most phase functions can be exactly represented under this form including the isotropic, linear anisotropic (also called Eddington), Rayleigh, hazy Mie and murky Mie phase functions [SH81]. The representation also allows exact formulations of implicitly energy-conserving phase functions expressed in terms of Legendre polynomials often used to model complex distributions such as that of Mie scattering. On the other hand, a few rational formulations such as the Henyey-Greenstein phase function are not exactly representable in the chosen basis and may only be approximated via an expansion into a series of Legendre polynomials or into a Taylor series. In the latter case, the chosen representation is still of a practical nature as described in the following subsections.

Due to the numerous advantages of the representation, we make the same assumption regarding the light distribution of order $N_I - 1$ and define $\varsigma = \cos(\vartheta)$ such that

$$I_c(\varsigma) = \sum_{n=0}^{N_I-1} c_I(n)\varsigma^n. \quad (10)$$

The resulting polynomial expression allows to exactly model both simple spotlights via a single term and complex sources with very few Legendre coefficients, and all the concepts subsequently discussed with respect to the phase function representation are equally applicable to the light source distribution.

The above expressions form the common formulation on which our solution is based as detailed in section 3.3. Besides discussing further the practical advantages of such representation, the following subsections additionally provide 2 methodologies for computing its coefficients.

3.2.1. Legendre Polynomials

Legendre polynomials are widely acknowledged as a standard basis to represent general phase functions and they are broadly used in light transport simulation [Cha60]. A phase function may be expressed as a series of such polynomials as follows

$$\Phi_c(\mu) = \frac{1}{4\pi}\sum_{n=0}^{N_\Phi-1} a_n P_n(\mu) \quad (11)$$

where the coefficients are defined by

$$a_n = \frac{2n+1}{2}\int_{-1}^{1} 4\pi\Phi_c(\mu)P_n(\mu)\mathrm{d}\mu. \quad (12)$$

The first coefficient evaluates to the integral of the representation over the whole spherical domain, and setting $a_0 = 1$ consequently guarantees normalization of the phase function. Moreover, the second coefficient relates to the asymmetry coefficient g as follows $a_1 = 3g$. Several phase functions may be expressed in this polynomial basis, either exactly with a finite number of terms, or approximately via truncation of the expansion into an infinite series. Coefficients for common scattering modes are reported in table 2 while expansions of the Mie phase function are provided in [CC55].

Table 2: *Coefficients of Legendre polynomials for several standard phase functions.*

Phase Function	Order	Coefficients
Isotropic	0	$a_0 = 1$
Linear Anisotropic	1	$a_0 = 1, -1 \leq a_1 \leq 1$
Rayleigh	2	$a_0 = 1, a_1 = 0, a_2 = \frac{1}{2}$
Henyey-Greenstein	∞	$a_n = (2n+1)g^n$

The coefficients $c_\Phi(n)$ can then be computed in-place from the coefficients a_n in order of increasing index n as follows

$$c_\Phi(n) = \frac{1}{4\pi} \sum_{\substack{k=n \\ k+=2}}^{\leq N_\Phi - 1} a_k \frac{(-1)^{\frac{k-n}{2}}}{2^k} \frac{(k+n)!}{\left(\frac{k-n}{2}\right)! \left(\frac{k+n}{2}\right)! \, n!}. \quad (13)$$

3.2.2. Taylor Series

In case a given phase function cannot be exactly expressed in the chosen polynomial basis, an alternative is to expand its formulation into a Taylor series representation. The expansion in the initial cosine space however presents numerous advantages over an expansion in the integration space as done in [PSP09a].

First, the expression of the successive derivatives is generally much simpler to evaluate, hence greatly reducing human burden, since the cosine parameter is considered as a single entity rather than as being itself a convoluted function of the variable of integration. In fact, the derivatives in the cosine space may even be described analytically such that the function can be approximated to an arbitrary precision by taking sufficiently many terms. For instance, we could easily derive such analytical expression for the Henyey-Greenstein phase function

$$\Phi_{HGc}^{(n)}(\mu) = \frac{1}{4\pi} \frac{1-g^2}{(1+g^2-2g\mu)^{\frac{3}{2}+n}} \left(\frac{g}{2}\right)^n \frac{(2n+1)!}{n!}. \quad (14)$$

The derivatives yield the coefficients of the truncated power series centered in μ_0. The polynomial coefficients can then be computed in-place from the latter in order of increasing index n by use of the binomial theorem to finally yield $c_\Phi(n)$ as follows

$$c_\Phi(n) = \frac{1}{n!} \sum_{k=0}^{N_\Phi - 1 - n} \Phi_c^{(k+n)}(\mu_0) \frac{(-\mu_0)^k}{k!}. \quad (15)$$

Moreover, because a series is expressed in terms of powers of the variable, the expansion typically converges rapidly whenever the magnitude of the latter is no greater than 1 and slowly otherwise. Given that the space of the variable μ is inherently bounded to $[-1,1]$, the formulation given in equation 9 consequently yields relatively quick convergence with only a few terms. This single formulation therefore naturally achieves the goal sought by the two complementary formulations of Pegoraro et al. [PSP09a].

3.3. Formulating the Medium Radiance

Substituting the expression of the cosine angle from equation 6 into the chosen angular representation given in equation 9, expanding the numerator using the binomial theorem and then rearranging the terms, the phase function formulation in the integration space reads

$$\Phi_c\left(-\frac{v^2-1}{v^2+1}\right) = \sum_{n=0}^{N_\Phi - 1} c_\Phi(n) \sum_{\substack{k=0 \\ k+=2}}^{2n} d_\Phi(n,k) \frac{v^k}{(v^2+1)^n} \quad (16)$$

where the coefficients are defined as

$$d_\Phi(n,k) = (-1)^{\frac{k}{2}} \binom{n}{\frac{k}{2}}. \quad (17)$$

Similarly, the expression of the light distribution from equation 10 becomes

$$I_c\left(\frac{d_{el}(v^2-1)+2d_c v}{v^2+1}\right) = \sum_{n=0}^{N_I - 1} c_I(n) \sum_{k=0}^{2n} d_I(n,k) \frac{v^k}{(v^2+1)^n} \quad (18)$$

where the coefficients read instead

$$d_I(n,k) = \sum_{\substack{l=k \bmod 2 \\ l+=2}}^{n-|n-k|} (-1)^{n-\frac{k+l}{2}} (2d_c)^l d_{el}^{n-l} \binom{n}{l} \binom{n-l}{\frac{k-l}{2}}. \quad (19)$$

The product of the phase function and light distribution may therefore be generally expressed under the form

$$\begin{aligned} &\Phi_c\left(-\frac{v^2-1}{v^2+1}\right) I_c\left(\frac{d_{el}(v^2-1)+2d_c v}{v^2+1}\right) \\ &= \sum_{n=0}^{N-1} c(n) \sum_{k=0}^{2n} d(n,k) \frac{v^k}{(v^2+1)^n} \end{aligned} \quad (20)$$

where $N = N_\Phi + N_I - 1$ and $c(n) = 1$ with

$$\begin{aligned} d(n,k) \;=\; & \sum_{m=\max\{0,n-N_I+1\}}^{\min\{N_\Phi-1,n\}} c_\Phi(m) c_I(n-m) \quad (21) \\ & \sum_{\substack{l=\max\{0,2\lceil\frac{k}{2}\rceil-2(n-m)\} \\ l+=2}}^{\min\{2m,2\lfloor\frac{k}{2}\rfloor\}} d_\Phi(m,l) d_I(n-m,k-l). \end{aligned}$$

Substituting equation 20 into equation 6 finally yields

$$\begin{aligned} L_m(x_a,x_b,\vec{\omega}) = \frac{\kappa_s}{h} e^{\kappa_t(x_a-x_h)} 2 \quad (22) \\ \sum_{n=0}^{N-1} c(n) \sum_{k=0}^{2n} d(n,k) \int_{v_a}^{v_b} \frac{e^{-Hv}}{(v^2+1)^{n+1}} v^k \mathrm{d}v \end{aligned}$$

and it consequently follows that the problem can be effectively reduced to the knowledge of a single antiderivative.

3.4. Solving the Indefinite Integral

Unlike the general mechanisms available for *derivatives*, systematic methods for determining the *antiderivative* of an arbitrary function unfortunately don't exist and each indefinite integral must consequently be treated individually, typically via human intervention. Although the logical steps of the analysis may be readily followed once a solution has been found for a given function, analyzing the problem and determining the actual mathematical reasoning leading to this solution is in general a rather non-trivial matter, and instances of known results are carefully compiled into comprehensive tables of integrals such as [GR07].

Since the previous antiderivative however appeared in none of the various compilations we consulted and is speculated to be currently unknown, we were led to investigate the problem on our own and the formulation of this new theoretical result is one of the fundamental contributions of our work. For $n \in \mathbb{N}$ and $m \in \mathbb{N}^*$, the antiderivative may be rewritten by factorizing the denominator as the product of terms of the imaginary entity $\imath^2 = -1$ using the identity $v^2+1 = v^2 - \imath^2$

$$\int \frac{e^{av}}{(v^2+1)^m} v^n \mathrm{d}v = \int \frac{e^{av}}{(v+\imath)^m (v-\imath)^m} v^n \mathrm{d}v. \tag{23}$$

Multiplying the numerator by $\frac{\imath}{2}\big((v-\imath)-(v+\imath)\big) = 1$ and simplifying subsequently yields

$$\int \frac{e^{av}}{(v^2+1)^m} v^n \mathrm{d}v = \frac{\imath}{2}\Bigg(\int \frac{e^{av}}{(v+\imath)^m (v-\imath)^{m-1}} v^n \mathrm{d}v - \int \frac{e^{av}}{(v+\imath)^{m-1} (v-\imath)^m} v^n \mathrm{d}v\Bigg). \tag{24}$$

Repeating the above process m times simultaneously for all summands then gives

$$\int \frac{e^{av}}{(v^2+1)^m} v^n \mathrm{d}v = \left(\frac{\imath}{2}\right)^m \sum_{k=0}^{m} (-1)^k \binom{m}{k} \int \frac{e^{av}}{(v+\imath)^{m-k} (v-\imath)^k} v^n \mathrm{d}v. \tag{25}$$

Recursively excluding from the process the terms whose exponent of either $v+\imath$ or $v-\imath$ is zero in the denominator while applying the previous factorization further to the remaining summands, and substituting $j = m-k$ subsequently leads to

$$\int \frac{e^{av}}{(v^2+1)^m} v^n \mathrm{d}v = \sum_{j=1}^{m} \left(\frac{\imath}{2}\right)^{2m-j} \binom{2m-1-j}{m-1} \left((-1)^{m-j} \int \frac{e^{av}}{(v+\imath)^j} v^n \mathrm{d}v + (-1)^m \int \frac{e^{av}}{(v-\imath)^j} v^n \mathrm{d}v\right). \tag{26}$$

Proceeding to a change of the variable of integration by defining $z = v+\imath$ or $z = v-\imath$ in each of the above integrals respectively, expanding the resulting n^{th}-order polynomials in the numerator using the binomial theorem, and rearranging the terms then yields

$$\int \frac{e^{av}}{(v^2+1)^m} v^n \mathrm{d}v = \sum_{j=1}^{m} \left(\frac{\imath}{2}\right)^{2m-j} \binom{2m-1-j}{m-1} \sum_{l=0}^{n} \binom{n}{l} \left((-1)^{m-j}(-\imath)^{n-l} e^{-\imath a} \int^{v+\imath} \frac{e^{az}}{z^j} z^l \mathrm{d}z + (-1)^m \imath^{n-l} e^{\imath a} \int^{v-\imath} \frac{e^{az}}{z^j} z^l \mathrm{d}z\right). \tag{27}$$

Splitting each of the above sums into 2 sub-sums containing only terms with either negative or positive exponents of the variable z, and substituting $k = j-l$ or $k = l-j$ respectively, subsequently leads to

$$\begin{aligned}\int \frac{e^{av}}{(v^2+1)^m} v^n \mathrm{d}v = \sum_{j=1}^{m} \left(\frac{\imath}{2}\right)^{2m-j} \binom{2m-1-j}{m-1} \Bigg(& \sum_{k=\max\{1,j-n\}}^{j} \binom{n}{j-k} \left((-1)^{m-j}(-\imath)^{n-j+k} e^{-\imath a} \int^{v+\imath} \frac{e^{az}}{z^k} \mathrm{d}z + (-1)^m \imath^{n-j+k} e^{\imath a} \int^{v-\imath} \frac{e^{az}}{z^k} \mathrm{d}z\right) \\ & + \sum_{k=0}^{\leq n-j} \binom{n}{j+k} \left((-1)^{m-j}(-\imath)^{n-j-k} e^{-\imath a} \int^{v+\imath} e^{az} z^k \mathrm{d}z + (-1)^m \imath^{n-j-k} e^{\imath a} \int^{v-\imath} e^{az} z^k \mathrm{d}z\right)\Bigg)\end{aligned} \tag{28}$$

The above reformulation ultimately expresses the antiderivative in terms of the following known indefinite integrals [GR07]

$$\begin{aligned}\int e^{av} v^n \mathrm{d}v &= \frac{e^{av}}{a} \sum_{i=0}^{n} \frac{n!}{(n-i)!} \frac{v^{n-i}}{(-a)^i} \\ \int \frac{e^{av}}{v^m} \mathrm{d}v &= \frac{a^{m-1}}{(m-1)!} \mathrm{Ei}(av) - \sum_{i=0}^{m-2} \frac{(m-2-i)!}{(m-1)!} \frac{e^{av} a^i}{v^{m-1-i}}.\end{aligned}$$

Finally substituting the above formulations, multiplying the denominator so as to remove the imaginary entity from the latter and expanding the corresponding factor in the numerator using the binomial theorem, the resulting expression may then be rearranged and simplified to yield our solution readily reported in equation 29 to follow, where the E function reads

$$\begin{aligned}E(a,v,j) &= \frac{1}{2}\left(\frac{\imath^j}{e^{\imath a}} \mathrm{Ei}(av+\imath a) + \frac{e^{\imath a}}{\imath^j} \mathrm{Ei}(av-\imath a)\right) \\ &= (-1)^{\lfloor \frac{j}{2} \rfloor} i_{(1-(j \bmod 2))}(a,v)\end{aligned} \tag{30}$$

with Ei being the complex-valued exponential integral function [AS72] whose real $\Re$ and imaginary $\Im$ parts define

$$\begin{aligned}i_0(a,v) &= \sin(a)\Re(\mathrm{Ei}(av+\imath a)) - \cos(a)\Im(\mathrm{Ei}(av+\imath a)) \\ i_1(a,v) &= \cos(a)\Re(\mathrm{Ei}(av+\imath a)) + \sin(a)\Im(\mathrm{Ei}(av+\imath a)).\end{aligned}$$

$$\begin{aligned}\int \frac{e^{av}}{(v^2+1)^m} v^n \mathrm{d}v = \frac{1}{2^{m-1}} \sum_{l=0}^{m-1} \frac{1}{2^l}\binom{m-1+l}{m-1}\Bigg(&\sum_{k=0}^{\min\{m-1-l,n\}}\binom{n}{k}\Bigg(\frac{a^{m-1-l-k}}{(m-1-l-k)!}E(a,v,m-n-l+k) \\ &- e^{av}\sum_{j=1}^{m-1-l-k}\frac{(j-1)!}{(m-1-l-k)!}\frac{a^{m-1-l-k-j}}{(v^2+1)^j}\sum_{\substack{i=(m-n-l+k-j) \bmod 2 \\ i+=2}}^{\le j}(-1)^{\frac{m-n-l+k-j+i}{2}}\binom{j}{i}v^i\Bigg) \\ &+\frac{e^{av}}{a}\sum_{k=0}^{\le n-m+l}\binom{n}{k}\sum_{j=0}^{n-m+l-k}\frac{(n-m+l-k)!}{j!}\frac{1}{(-a)^{n-m+l-k-j}}\sum_{\substack{i=(-m+l+k-j) \bmod 2 \\ i+=2}}^{\le j}(-1)^{\frac{-m+l+k-j+i}{2}}\binom{j}{i}v^i\Bigg)\end{aligned} \tag{29}$$

4. Implementation

While the mathematics underlying the derivation might seem daunting at first, implementing the method really only entails literally converting the few relevant equations into code, and figure 2 provides an algorithm outline in order to help identifying those involved in that process. Although the exponential integral is part of the commonly employed Boost C++ libraries, we used instead the implementation provided in [PSP09a] in our experiments to ease portability onto graphics hardware.

With respect to efficiency, it is worth observing that redundant computation may be easily avoided by evaluating the terms i_0 and i_1 in equation 30 only once before entering the loops as both are independent of the value of the iterators from equation 29. Similarly, the various power and factorial terms as well as the binomial coefficients involved can be incrementally computed via pre-iterative initialization of the variables which may then be updated in constant time within the loops. While doing so, the second and third lines of equation 29 might resort to basic manipulations of the type $\sum_j^m \frac{j!}{m!}a^{m-j} = \frac{a^m}{m!}\sum_j^m \frac{j!}{a^j}$ allowing the external factor to be ultimately generated as a by-product of the iterative process. Finally, although the following observation does not hold true in arbitrary settings, our experiments have empirically shown that, when solving the air-light integral, the sum on the third line of equation 29 systematically evaluates to zero, which suggests additional potential for optimization.

ComputeEyeRadiance()

1. Compute L_m;
2. Compute bounds of integral as in equations 5 and 7;
3. Compute coefficients $c_{\Phi|I}(n)$ as in equation 13 or 15;
4. Compute L_m as in equation 22;
5. Compute coefficient $d(n,k)$ as in equation 17/19/21;
6. Compute integral using equations 29 and 30;
7. Compute L_r as in equations 2 and 8;
8. Compute L as in equation 1;

Figure 2: *Algorithm outline summarizing the equations involved in implementing the method.*

5. Results

The accuracy of the method was first assessed via a CPU implementation facilitating quality evaluation in comparison to that of previous approaches. Figure 3 illustrates the results obtained over a wide range of ray trajectories using an environment camera while rendering high/low-density haze with a Rayleigh phase function. In such setting, Lecocq's 3-term expansion [BAM06] yields strong ghosting artifacts in light-backfacing directions while overall inaccuracies may be most easily noticed in the skyline. The latter directions are also problematic for Sun's 512×512 precomputed tables which lead to visual artifacts (see insets) divulging bilinear interpolation errors in addition to those most prominent in high-density media caused by extrapolation. In contrast, our closed-form solution faithfully matches the reference image computed offline using Monte Carlo estimation, as emphasized in figure 5.

Figure 4 depicts the results obtained with relatively focused spotlight distributions. Using the first 4 terms of the Taylor series as reported in [PSP09a], Pegoraro's dual-formulation leads to inaccuracies in the glows of the 2 leftmost light sources and to erroneous artifacts below the rightmost one where the approximation results in negative estimates of the positive integral. Although the artifacts might be reduced by hand-coding additional terms in the series whose lengths here become quickly intractable, inaccuracies still remain even with 6 terms in the Taylor expansions. On the other hand, our results here again match the reference solution exactly as highlighted in figure 5.

In addition, figure 6 shows a complex anisotropic light source distribution easily modeled using only 2 Legendre coefficients. Using our method, the solution can be efficiently rendered in closed-form, whereas none of the previous approaches could conceivably handle such a case even semi-analytically. We also wish to emphasize that all the results presented in this paper use exact representations of the various angular distributions, and details about their expressions may be found in table 3.

The performance characteristics of our closed-form solution compared to that of previous approaches were also evaluated via a Cg-based GPU implementation running on an

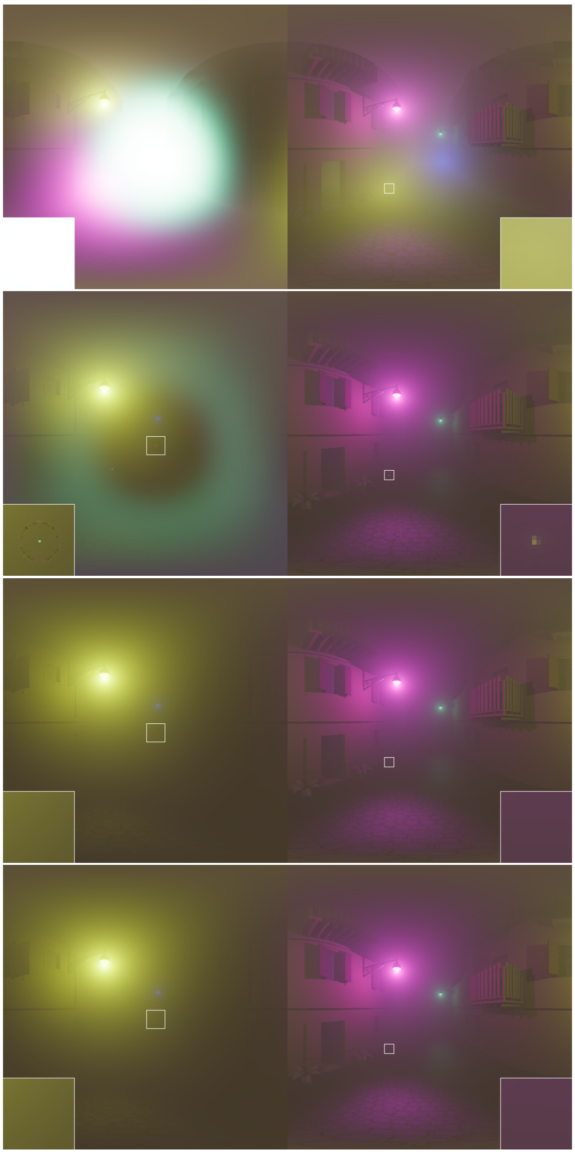

Figure 3: *An alley covered in respectively thick (left half) and thin (right half) anisotropic haze lit by 4 colorful streetlamps, rendered using (from top to bottom) Lecocq's method, Sun's method, our closed-form solution, and offline Monte Carlo estimation.*

NVIDIA GeForce GTX 280 under Windows XP 64-bit. The calculation was carried independently for each color channel so as to support chromatic effects, and while the computational cost increases with the order of the integrand, interactivity is still maintained for various common scattering modes as illustrated in table 4 which reports the frame rates benchmarked on a low-geometry scene so as to predominantly measure the cost of the air-light integration schemes.

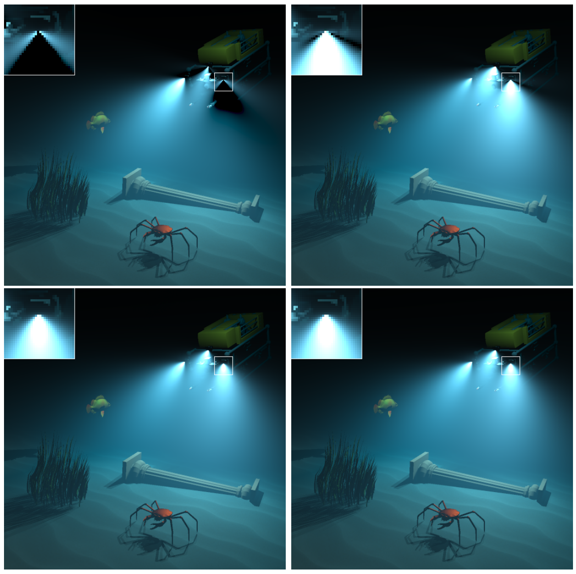

Figure 4: *An underwater scene illuminated by the spotlights of a submersible, rendered using (top-left) Pegoraro's method with 4 terms, (top-right) Pegoraro's method with 6 terms, (bottom-left) our closed-form solution, and (bottom-right) offline Monte Carlo estimation.*

Table 3: *Number and order of the terms constituting the exact expressions of the various anisotropic functions used in the respective figures.*

Function	Terms	Orders	Figure
Isotropic	1	0	
Linear Anisotropic	2	0, 1	
Rayleigh	2	0, 2	3
Spotlight	1	10	4
Light Ball	6	0, 2, 4, 6, 8, 10	6

Due to the simplifications that they rely on, we acknowledge that previous approaches are admittedly more suited to time-critical applications whenever visible artifacts are tolerable. Nevertheless, it is important to recall that unlike previous techniques which *approximate* the solution for fairly limited cases such as isotropic light sources [SRNN05] or low-frequency distributions [PSP09a], our result ***is*** the mathematical solution to an actually much broader set of configurations including complex phase functions and light sources, and it can therefore be used to effectively assess the quality of the aforementioned inexact methods.

6. Discussion and Future Work

Thanks to their generality, numerical methods can most often implicitly account for visibility of the light sources along view rays as part of the integrand, although doing so is inher-

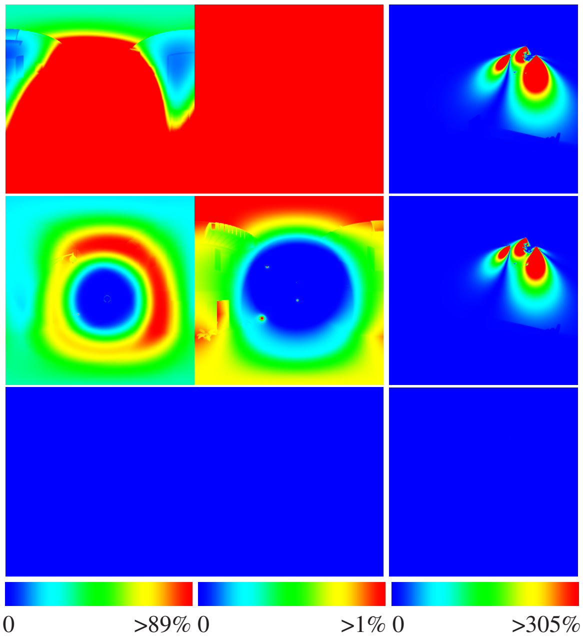

Figure 5: *Color-coded visualization of the absolute error of the results relative to the average image intensity (left) from figure 3 for (from top to bottom) Lecocq's method, Sun's method, and our closed-form solution, and (right) from figure 4 for (from top to bottom) Pegoraro's method with 4 and 6 terms respectively, and our closed-form solution.*

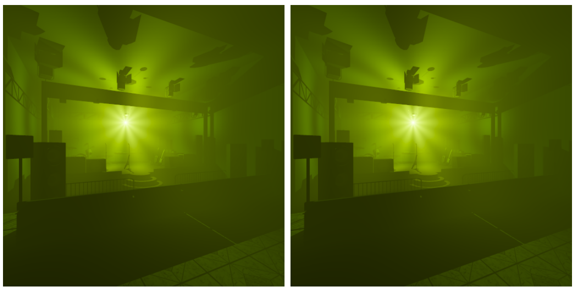

Figure 6: *A concert stage lit by a high-frequency anisotropic light ball rendered analytically using our closed-form solution (left) and compared against the reference image (right).*

ently prone to under-sampling artifacts. In contrast, analytical approaches, including those from previous work as well as the one presented in this paper, typically need to explicitly handle the discontinuities due to volumetric shadows by partitioning the domain of integration so as to exclude occluded intervals from the integral as done in [Jam03] or [BAM06]. Despite the extra computational cost entailed by the shadow-volumes algorithm, coupling our integration scheme with the latter would consequently provide a simple means of effectively determining exact shadow boundaries.

Table 4: *Frame rates achieved with our closed-form solution on graphics hardware compared to that of previous approximate methods at a resolution of* 768×768 *for various chromatic scattering modes. Both anisotropic light distributions are of order 4, and the performance for Pegoraro's dual-formulation is reported for 4 and 6 terms in the Taylor series expansion respectively.*

FPS	Lecocq	Sun	Pegoraro	Our method
Isotropic	932	998		315
Linear Anisotropic		984		80.7
Rayleigh	991	988		44.7
Spotlight			84.9/31.7	3.53
Light Ball				1.71

Moreover, the latter performance characteristics actually illustrate the main practical limitation of our solution, namely, its computational cost which is a supra-linear function of the order of the angular distributions. While the approach performs reasonably well for several common scattering modes, interactive frames rates may currently not be achieved for high-order representations and future directions of research could consequently investigate alternatives to improve the efficiency of the evaluation scheme.

On the other hand, the technique does scale linearly in the number of light sources, and area lights may be easily represented as a set of several anisotropic point sources. Also, while almost all naturally occurring phase functions are rotationally symmetrical about the incident direction, the 1-D assumption does not hold when dealing with light sources parameterized by both polar and azimuthal angles. Generalizations of the method could therefore consider 2-dimensional light distributions in order to make the solution applicable to a broader range of configurations. Finally, our analytical approach opens an avenue to many extensions, including the derivation of closed-form solutions for non-punctual light sources and inhomogeneous media as well as higher-order scattering events.

7. Conclusion

In this paper, we have presented the very first closed-form solution to the air-light integral in homogeneous media for general 1-D phase functions and punctual light sources. Assuming a generic representation of angular distributions, we have provided a mathematical formulation of the problem and shown that the latter effectively reduces to the knowledge of a single indefinite integral. In order to determine the antiderivative of interest, we have then introduced a novel analytical derivation leading to the formulation of a closed-form solution which, to the best of our knowledge, has never appeared before in the literature.

This new theoretical result enables the analytical computation of exact solutions to complex scattering phenomena which, up until now, still represented an open problem in the overall light transport simulation community. The technique makes the calculation of ground truth solutions in a finite number of computational steps finally possible, and therefore the efficient generation of reference images to evaluate the quality of the many approximations to the single-scattering model that have been proposed in the past. Furthermore, it also allows high-quality results to be achieved with semi-interactive performance on current-generation graphics hardware for several common scattering modes. Finally, although the solution is currently bound to the assumption of single scattering, homogeneity of the medium and punctuality of the light sources, we believe that it represents a major step forward towards deriving more generic analytical solutions to light transport and hope it will stimulate subsequent research in the field.

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