An In-depth Investigation of the Multigrid Approach to Steady and Transient EHL Problems

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Multigrid methods have proved robust and highly desirable in terms of the iteration speed in solving elastohydrodynamic lubrication (EHL) problems. Lubrecht, Venner and Ehret, amongst others, have shown that multigrid can be successfully used to obtain converged solutions for steady problems. steady problems.

A detailed study reinforces these results but also shows, in some cases, that while multigrid techniques give initial rapid convergence, the residuals - having dropped to a low level - may reach a stalling point, mainly due to the cavitation region. The study will explain this behaviour in terms of the iterative scheme and show how, if this happens, the errors in the fine grid solution can be reduced further. Example results of both steady and transient EHL problems (including a thermal viscoelastic case) are shown with further developments into adaptive meshes considered.

1. Introduction

Elastohydrodynamic lubrication (EHL) problems have been solved numerically since Petrusevich's first results were published in 1951 [1]. Almost half a century later, it is now possible to solve large, complicated problems routinely. The industrial requirement is for accurate solutions as quickly as possible to help in the development of new lubricants and model the behaviour of the components they separate. For example, it is not unusual to be testing several oils under a variety of loads at a variety of speeds. One of the most successful methods that has been employed to speed up EHL codes is that of Multigrid. Since it was first used by Lubrecht for both line [2] and point [3] contacts, it has become a standard tool for fast solutions of detailed problems.

The aim of this paper is to investigate the detailed performance of multigrid methods for EHL problems and in particular to assess and improve the convergence behaviour as far as possible.

The rest of the paper is laid out as follows. In Section 3 we present the governing equations of the problem of a circular point contact using the notation from Section 2. Section 4 briefly describes the multigrid method, as well as the mechanics of the solver used. Section 5 uses a transient flow reversal case to illustrate the problems we attempt to remedy. Sections 6 and 7 describe possible difficulties we have encountered with convergence, and they present our methods for addressing these issues. The paper is concluded in Section 8 where we also look forward to future work using adaptive meshes.

2. Notation

a	halfwidth of Hertzian contact
H	non-dimensionalised film thickness
H_{00}	film thickness central offset
p	pressure
p_0	ambient pressure
p_h	maximum Herzian pressure
P	non-dimensionalised pressure: $P = p/p_h$

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R_x	reduced radius of curvature
T	non-dimensionalised time
u_s	sum of velocities of contacts
X	dimensionless coordinate
Y	dimensionless coordinate
z	viscosity index
α	pressure viscosity index
ϵ	coefficient in Reynolds equation
λ	coefficient in Reynolds equation
η	viscosity
η_0	viscosity at ambient pressure
$\bar{\eta}$	non-dimensionalised viscosity: $\bar{\eta} = \eta/\eta_0$
ρ	density
ρ_0	density at ambient pressure
$\bar{\rho}$	non-dimensionalised density: $\bar{\rho} = \rho/\rho_0$

3. Governing Equations

The EHL problem is governed by a system of equations which both define the pressure distribution due to the lubricant thickness, and the deformation of the surfaces due to the pressure. For circular point contact cases, as used in this paper, the equations in non-dimensionalised form, are: the Reynolds equation:

$$\frac{\partial}{\partial X}\left(\epsilon\frac{\partial P}{\partial X}\right) + \frac{\partial}{\partial Y}\left(\epsilon\frac{\partial P}{\partial Y}\right) - \frac{u_s}{u_s(0)}\frac{\partial(\bar{\rho}H)}{\partial X} - \frac{\partial(\bar{\rho}H)}{\partial T} = 0, \tag{1}$$

the film thickness equation:

$$H(X,Y) = H_{00} + \frac{X^2}{2} + \frac{Y^2}{2} + \frac{2}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{P(X',Y')\,dX'\,dY'}{\sqrt{(X-X')^2+(Y-Y')^2}}, \tag{2}$$

and the force balance equation:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(X,Y)\,dX\,dY = \frac{2\pi}{3}, \tag{3}$$

where ϵ and λ are given by

$$\epsilon = \frac{\bar{\rho}H^3}{\bar{\eta}\lambda u_s(0)}, \tag{4}$$

and

$$\lambda = \frac{6\,\eta_0\,R_x^2}{a^3\,p_h}. \tag{5}$$

Equation (2) is the most computationally intensive part of the solution process. This is because, in a discretised form, each point is a function of all the pressures at all the points in the computational domain. Multi-level multi-integration, as first proposed for EHL problems by Brandt and Lubrecht [17], and Venner [4], has speeded this up dramatically, reducing the amount of work in this calculation from $O(n^2)$ to $O(n \ln n)$, where n is the total number of points in the domain.

For completion, the model used for viscosity is derived from the Roelands equation [5],

$$\eta(p) = \eta_0\, exp\left\{\frac{\alpha\,p_0}{z}\left[-1+\left(1+\frac{p}{p_0}\right)^z\right]\right\} \tag{6}$$

and for density the Dowson and Higgison relation [6] is employed:

$$\rho(p) = \rho_0\left(1+\frac{5.8\times10^{-10}\,p}{1+1.7\times10^{-9}\,p}\right). \tag{7}$$

4. The Multigrid Solver

To solve the problem we, first, spatially discretise the equations on a regular grid of size 2^k+1 x 2^k+1 points. We can then use symmetry along the $y = 0$ centreline to halve the size of the domain.

The smoothing process is defined in the following stages. First, the Reynolds Equation (1) for the pressure is solved as described below. After that is done the force balance equation (3) is used to relax the value of H_{00} before it is used in the calculation of the new film thickness in (2). Finally, the viscosity and density are updated. This is implemented in software built upon the FDMG code [7] and is described in [8].

In a traditional non-multigrid framework this smoothing process is repeated ad infinitum on the grid described above. The advantage of multigrid is that it can start on a much coarser grid (ie where k is lower). The initial smoothing steps

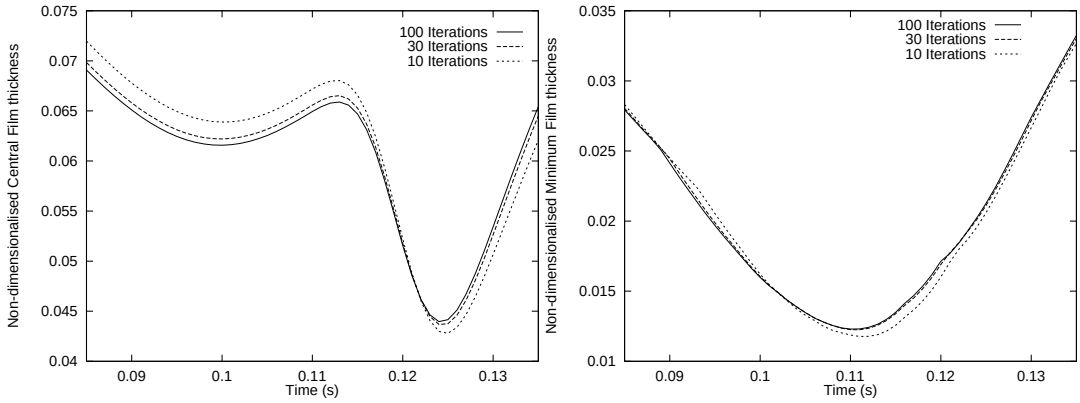

Figure 1. Central and minimum film thicknesses during reversal of entrainment with 10, 30 and 100 iterations per timestep.

are performed before stepping up to the next finer grid in a hierarchical manner. Elimination of large errors in the initial Hertzian solution, as well as very low frequency errors, are achieved in this way. These issues are explained in detail in, for example, [9] and [10], along with more information on multigrid methods and strategies.

The principle is relatively straightforward. Each level in the hierarchy of grids is regularly meshed so that in each coarser-finer pair of grids all the coarser points are coincident with points on the finer mesh, with an extra point midway between each two adjacent coincident points. The simplest multigrid cycle is the V-cycle which starts on the finest grid, smooths the solution and then coarsens both the solution and the residual to the next grid where it again smooths. This process is repeated until the coarsest grid is reached where, after smoothing, a prolongation operator is used to then update the solution on the next grid up. Smoothing is then applied. When we have smoothed on the finest grid we are said to have completed one multigrid cycle. The operations of prolonging and coarsening between grids are defined in terms of stencils showing how the interpolation of the local values are used to calculate the coarser and finer representations of the solution. We can define them as operators on a solution by

$$\tilde{\underline{u}}^{k-1} = I_k^{k-1}\underline{u}^k \tag{8}$$

for coarsening from grid k to grid k-1, and for prolonging back by

$$\tilde{\underline{u}}^{k} = I_{k-1}^{k}\tilde{\underline{u}}^{k-1}. \tag{9}$$

The nature of the EHL problem means that across the domain there are two very different regions. In the contact region (around (0,0)) the Reynolds equation in the transient case is hyperbolic in character, whereas in the rest of the domain it is parabolic. Different solution schemes are needed in each of these regions. There are a variety of possible schemes that are commonly employed in these areas. These are summarised and compared in papers such as [11] and [12]. It is also important to note that the far right-hand side of the domain is a cavitation region where the Reynolds equation no longer applies.

5. Transient Problems

One of the current areas of particular interest is solving transient problems. These require a robust strategy for quickly obtaining converged

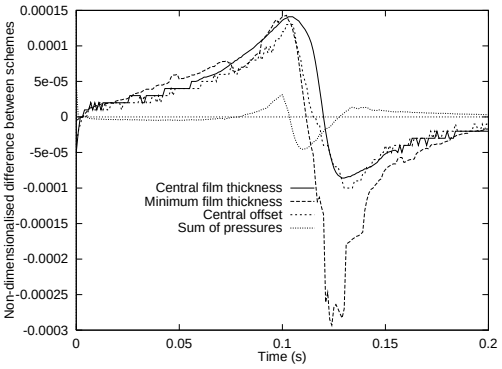

Figure 2. Difference in results between distributed and Nurgat's non-distributed scheme for a reversal problem over 0.2s.

results at each time step. Each timestep is a separate problem, but excessive errors must not be allowed to propagate through from one step to the next. Figure 1 shows the effect of timestepping before all the variables have fully converged with the three curves showing 10, 30 and 100 fine grid iterations per timestep. This example is from [13] and shows the central and minimum film thicknesses during linear reversal of entrainment from 0.1ms^{-1} to -0.1ms^{-1} in 0.2s.

The variation between the curves in Figure 1 clearly illustrates how the solution after 10 iterations was not fully converged. In this case we have used timesteps of 0.001s which gave 100 timesteps each side of the point of reversal. A successful transient code needs to be able to quickly compute the next solution but there is a delay in the convergence of the results.

The iteration method used is that of Nurgat, Berzins and Scales [12]. This offers an alternative to the distributed scheme of Venner [4] and has since been compared to the Venner scheme on a number of highly loaded cases. The results and convergence times were similar between the two methods. This is illustrated in Figure 2 which shows the difference in solution of the results obtained for the 30 iteration case between the distributed scheme and that of Nurgat for central and minimum film thicknesses, the central offset value, H_{00} and the sum of pressures. These difference are small and well within the errors shown above for non-converged results.

6. Multigrid Convergence and Stalling

The essence of multigrid is the use of coarser grids to correct the solution quickly by eliminating errors of frequencies that would be expensive to remove on the fine grid. The EHL problem is highly non-linear and so we need to use the Full Approximation Scheme (FAS). This means that, rather than just transferring the error between grids to correct the solution, we transfer the solution as well. FAS accomplishes this by coarsening the previous fine solution, smoothing on the coarser grid and then prolonging a correction back to the finer grid. This is called Coarse Grid Correction (CGC).

In EHL problems the coarse mesh solution may be inherently different to the fine mesh solution on the edge of the cavitation region because the position of the free boundary may move half a coarse mesh cell (one fine cell). This means that when interpolating back, the new solution is introducing an error at this boundary. This is shown in Figure 3 which shows the residual levels across the half domain at two separate stages in the solution process - both after returning from the CGC process. The more prominent, faint surface shows the early stages of convergence where residual levels across the whole domain are noticable. The lower, darker surface shows that most of the error has been smoothed away except that exactly the same error is reappearing on the cavitation boundary. This error is then smoothed away on the fine grid but is reintroduced the next time the CGC is made. This means there is a stage when the errors smoothed away on the finest grid are equally balanced by the errors added to the solution by the CGC process.

This stalling point is illustrated in Figure 4 which shows, in a relatively highly loaded case, the way the residuals on the finest grid reach a

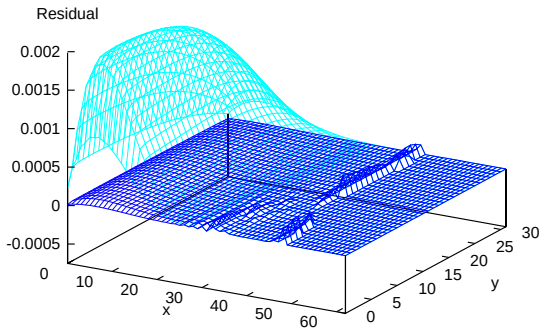

Figure 3. Residual levels across the half domain showing errors on the cavitation boundary are not reduced.

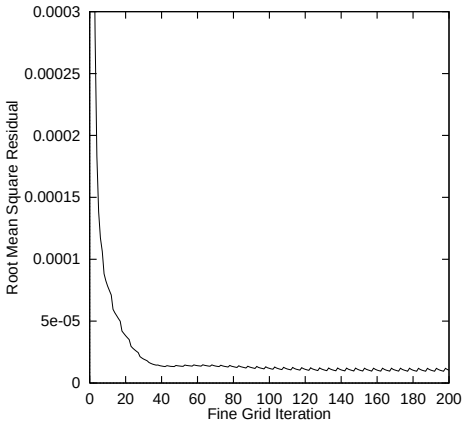

Figure 4. Residual values reaching a stalling point in a multigrid solution method.

point beyond which further multigrid cycles have no effect.

Careful treatment of this free boundary is therefore necessary. In [4] it says that "near the cavitational boundary both for the transfer of the residuals as well as for the transfer of the solution to the coarse grid *injection* should be used." In our experience it may be necessary also to fix the free boundary on coarser grids to ensure that, when it is prolonged back it has not moved.

The EHL problem has the added complications of the sharp pressure rise on the edge of the contact region (where ϵ changes by many orders of magnitude) and the pressure ridge. The advantage of the multigrid start to performance is enormous and the use of standard multigrid techniques for initial error reduction is also without question. However, when accurate, fully converged solutions are required a good strategy is to remove the coarsest grid from use at the stage when the CGC is no longer useful. For full convergence to occur, this may or may not lead to a stage where just the fine grid remains.

The decision of when to remove grid levels is determined by a number of factors, the most obvious one being the ratio of the reduction in size of the residual from the CGC process to the reduction that would have been expected by smoothing on the fine grid alone. To simply say that there is an advantage is not enough because we must also include whether the time spent doing work on the coarser mesh is bringing enough of an error reduction on the finer mesh to justify using it.

An example of this can be seen in the following example from [14]. This is a thermal viscoelastic case with sliding. Running it on a series of three grids (smoothest 65x65) with and without our new method gave results as shown in Figure 5. We can see that the residual level using the new method has not stalled and that convergence is linear. In this case only the coarsest grid has been discarded (after around 300 fine grid iterations) and so multigrid is still being used all the way to machine accuracy.

An important point is that for the performance of the code still to be efficient we need to

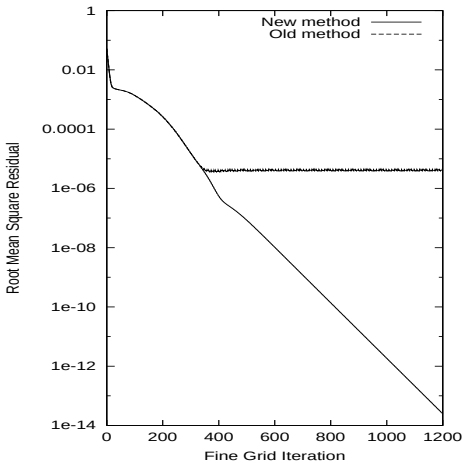

Figure 5. Effectiveness of multigrid cycles in reducing the residual in a thermal viscoelastic example with sliding.

continue to use multi-integration up to our original coarse level. This operation is not dependent on moving between grids, just using the values of pressure at the coincident points. If there is no decrease in convergence rate when grids are discarded (up until just the fine grid remains) then there will be a (small) work reduction.

7. The Force Balance Equation

Although the convergence of the Reynolds Equation (1) is of primary importance to the convergence of the system, care must be taken to ensure that the Force Balance Equation (3) is satisfied to a similar degree of accuracy. The method we are using is that described by Venner [4].

$$H_{00} \leftarrow H_{00} - c\left(\frac{2\pi}{3} - h_x h_y \sum_{i=1}^{n_x}\sum_{j=1}^{n_y} P_{i,j}\right) \qquad (10)$$

In this method the force balance equation is relaxed only on the coarsest grid and corrections made to be used on the finer grids. We have seen above that the solution on this grid is different to the solution on finer grids and hence the converged value of H_{00} will differ between grids. Table 1 highlights the variation in the converged value of H_{00} on different grids for a typical problem. From equation (10) it is clear that if H_{00} is not correct, then the sum of the pressure is not correct, meaning we have only strictly satisfied two of our three equations.

Table 1
Variation of non-dimensionalised H_{00} between grids.

Grid dimensions			H_{00}
17	x	17	-1.0504
33	x	33	-0.9508
65	x	65	-0.9286
129	x	129	-0.9243

There are several different approaches possible for allowing H_{00} to vary over the grids. These include using one independent value on each grid, passing the value between grids, or only passing the value up/down through the grids. These stategies are summarised in Table 2. In our computational experience the quickest, most accurate strategy for fine grid Force Balance Equation convergence is passing the value of H_{00} from the finest grid down to coarser one but not back up.

In equation (10), the parameter c is a small enough relaxation factor (eg c=0.01) to avoid unstable oscillations [4]. Again, during the initial convergence of the solution this is very useful but a stage is reached where this contribution is too small.

Experiments have shown us that the the residuals will decay to machine accuracy as the sum of pressures converges to $\frac{2\pi}{3}$ and hence H_{00} converges on its final value. After the initial convergence has been achieved it is possible to increase the value of c in (10). A similar modification to the relaxation value in the chosen scheme for solving the Reynolds equation 1 is also beneficial. These changes were used in the generation of Figure 5.

Table 2
Strategies for passing H_{00} between grids.

Strategy for updating H_{00}	H_{00} when 'converged' on finest grid	Sum of pressures ($\frac{2\pi}{3} \approx 2.0944$)
Calculate on coarsest grid only	-0.8950	1.9903
Calculating on each grid:-		
Keep each grid independent of others	-0.9293	2.1005
Pass one value freely up and down grids	-0.9181	2.0597
Pass coarsest value up to finest	-0.8805	1.9726
Pass finest value down to coarsest	-0.9299	2.0934

8. Conclusions and Future Work

The combination of multigrid and multi-level multi-integration techniques produce fast solutions to EHL problems. We have presented three separate amendments to the strategy previously employed. These each produce more accurate results with less work than was previously necessary. We have shown that the Force Balance Equation must not be treated as a side issue but an overriding factor governing the final convergence of results. The overall convergence of our new approach can be seen in Figure 5.

We have now overcome many of the problems discussed in [12] concerning "delivering better convergence in terms of the residual" in addition to "reducing [the] CPU time" although we will endeavour to continue to make improvements.

The next stage of our research will be to investigate the need to completely mesh the entire computational domain using the finest level of refinement. The region of greatest interest is where the second derivative terms are highly variable, which in EHL problems is the contact region, in particular around the pressure ridge near the cavitation region. The option of not having as fine a mesh outside the regions of interest will significantly reduce the work per fine grid level - especially as the surface will be almost undeformed and hence the film thickness will be approaching parabolic. During the preparation of this paper we learned of other work using adaptive meshes for EHL problems [15].

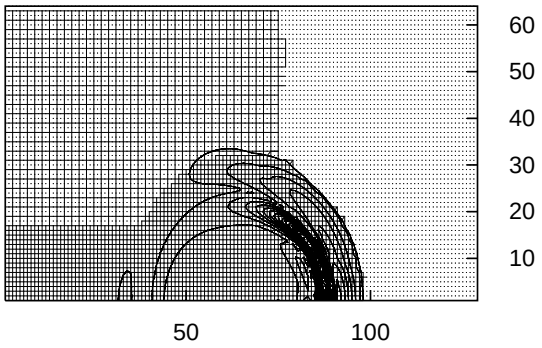

Figure 6. Use of adaptive gridding on a half domain, with second derivative contours overlayed.

Initial work has shown a saving of at least a third on the overall time taken for the calculation. Figure 6 shows a two-level mesh containing both 129x129 and 65x65 resolution regions. The dots are fine mesh nodes not calculated upon. The contours represent the change in value of $\left|\frac{d^2P}{dx^2}\right| + \left|\frac{d^2P}{dy^2}\right|$. The use of monitor functions, such as these contours, to guide mesh refinement is common in the adaptive mesh community. This approach has the important advantage of focussing the computational work where it is needed most. An obvious extension is to

combine such ideas with accurate time stepping approaches.

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