A SOLUTION BASED H^1 NORM TRIANGULAR MESH QUALITY INDICATOR.

M. BERZINS*

Abstract. The issue of mesh quality measures for triangular (and tetrahedral) meshes is considered. A new mesh quality measure is based both on geometrical and solution information and is derived by considering the error in the H^1 norm when linear triangular elements are used to approximate a quadratic function. The new measure is then compared with the recent mesh quality measure based on the L_2 norm. Simple examples are used to show that the choice of norm is critical in deciding what is a good triangulation.

Key words. Unstructured meshes, triangular mesh quality, H^1 error information.

AMS(MOS) subject classifications. 65N30, 65N50 AMSMOS}

1. Introduction. The increasing use of p.d.e. solvers based on triangular and tetrahedral meshes e.g [4] [14] raises the issue of whether the mesh is appropriate to represent the solution. The best approach is to have computable error estimates for each solution component. If such estimates are not available then the approach often taken is to view mesh quality as being independent of the solution, [5,9]. A number of authors have shown that it is both the shape of the elements and the local solution behaviour that is important, particularly for highly directional flow problems see [12,13] and the analysis of Babuska and Aziz [2], who showed that the requirement for triangles was that there should be no large angles.

This work has motivated recent work [3] in which a new mesh quality indicator is derived. This indicator is based on the L_2 norm of the error in a linear triangular or tetrahedral element approximation of a quadratic function. The indicator has the advantage that both solution and geometry based information is used to assess the quality of the mesh. One possible disadvantage of this indicator is that in many finite element calculations the error is estimated in the H^1 norm and as Dupont has noted, [6], there are situations in which the convergence is not obtained in the H^1 norm but is obtained in the L_2 norm.

The aim in this paper is thus to derive an H^1 based mesh quality indicator and to compare it with the L_2 norm based indicator of Berzins [3]. The existing work will de summarised in Sections 2 and 3 of this paper. The new indicator will be derived in Section 4 and applied to two test cases in Section 5 . Thes results will show that the situation can be worse than that suggested by Babuska and Aziz [2] and that a good triangulation may depend critically on the choice of norm. The paper closes with the derivation of a simple technique based on derivative jumps across edges to calculate the quantities used in the mesh quality indicators when a non-quadratic solution is approximated by linear elements.

2. Nadler's Error Estimate for Triangles. The starting point for the derivation of a new mesh quality indicator is the work of Nadler [10] who derives a particularly appropriate expression for the interpolation error when a quadratic function is approximated by a piecewise linear function on a triangle. Consider the triangle T defined by the vertices v_1, v_2 and v_3 as shown in Figure 2.1 below. Let h_i be the length of the edge connecting v_i and v_{i+1} where $v_4 = v_1$.

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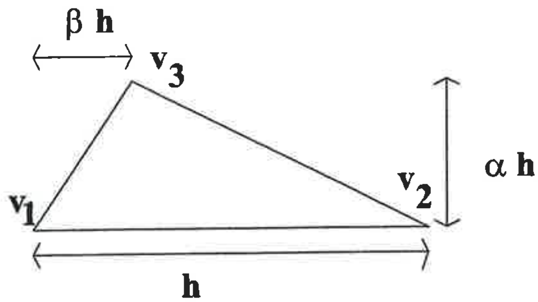

FIG. 2.1. *Babuska and Aziz Example Triangles.*

Nadler [10] considers the case in which a quadratic function

$$u(x,y) \;=\; \frac{1}{2}\underline{x}^T\; H\; \underline{x} \; where \;\; \underline{x} \;=\; \begin{bmatrix} x \\ y \end{bmatrix} \tag{2.1}$$

is approximated by a linear function $u_{lin}(x,y)$, as defined by linear interpolation based on the values of u at the vertices. Denote the error by

$$e_{lin}(x,y) = u_{lin}(x,y) \;-\; u(x,y) \tag{2.2}$$

Nadler, [10] as quoted in Rippa [13], shows that

$$\int_T (e_{lin}(x,y))^2 dx\; dy \;=\; \frac{A}{180}\left[((d_1 + d_2 + d_3)^2 + {d_1}^2 + d_2^2 + d_3^3\right] \tag{2.3}$$

where A is the area of the triangle and $d_i = \frac{1}{2}(v_{i+1} - v_i)^T\; H\; (v_{i+1} - v_i)$ is the derivative along the edge connecting v_i and v_{i+1}.

Example 1 In the case when the matrix H is positive definite with diagonal entries p^2 and q^2 and symmetric off-diagonal entries pq then

$$d_i = (p\;\Delta x_i \;+\; q\;\Delta y_i)^2 \;\; where\;\; v_{i+1} - v_i = [\Delta x_i, \Delta y_i]^T$$

In the case of the triangle in Figure 1 assuming that x and y are in the horizontal and vertical directions respectively, the values of d_i are $d_1 = p^2 h^2$, $d_2 = h^2(-(1-\beta)p + \alpha q)^2$ and $d_3 = h^2(\beta p - \alpha q)^2$.

3. An L_2 Mesh Quality Indicator for Linear Triangular Elements. In this section a summary will be given of the mesh quality indicator of Berzins [3] which is based on the work of Nadler [10]. This indicator takes into account both the geometry and the solution behaviour. The starting point for this indicator is equation (2.3): in the case when the values of d_i are all equal then each edge makes an equal contribution to the error. However in order to take into account in a consistent way the fact that the values of d_i may be of different signs it is necessary to consider their absolute values. With this in mind, the scaled form of the derivatives d_i are defined by

$$\tilde{d}_i = \frac{|d_i|}{d_{max}} \; where\; d_{max} \;=\; max\,[|d_1|, |d_2|, |d_3|] \tag{3.1}$$

For notational convenience define

$$\tilde{q}(\underline{\tilde{d}}) \;=\; (\tilde{d}_1 + \tilde{d}_2 + \tilde{d}_3)^2 + \tilde{d}_1^2 + \tilde{d}_2^2 + \tilde{d}_3^2 \tag{3.2}$$

where $\underline{\tilde{d}} = [\tilde{d}_1, \tilde{d}_2, \tilde{d}_3]^T$. A measure of the anisotropy in the derivative contributions to the error is then provided by

$$q_{aniso} \;=\; \frac{\tilde{q}(\underline{\tilde{d}})}{12} \tag{3.3}$$

The definitions of the coefficients $\tilde{d}_i$ in equation (3.1) results in the bounds

$$1/6 \ \leq \ q_{aniso} \ \leq \ 1 \tag{3.4}$$

Consider a triangle with only one edge contributing to the error. In this case $q_{aniso} = 1/6$ whereas if two edges contribute equally and the third makes no contribution $q_{aniso} = \frac{1}{2}$. A consistent and related indicator based on geometry alone is given by [3]

$$q_m(\underline{h}) = \frac{\tilde{q}(\underline{h})}{16\sqrt{3}\ A}$$

where $\underline{h} = [h_1, h_2, h_3]^T$, has value 1 for an equilateral triangle and tends to the value infinity as the area of a triangle tends to zero but at least one of its sides is constant. It is easy to show, [3], that this is a linear combination of those of Bank [2] and Weatherill [14]. The relationship between q_{aniso} and the linear interpolation error is that when the matrix H is positive definite, i.e. $d_i > 0$, then

$$q_{aniso} = \frac{15}{A\ d_{max}^2} \int_T (e_{lin}(x,y))^2 dx\ dy, \tag{3.5}$$

thus showing that the indicator is a scaled form of the interpolation error in this special case.

4. Error Estimation in the H^1 Norm. The extension of to the case of the H^1 norm is achieved by considering the case in which a quadratic function in equation (2.1) is approximated by a linear function $u_{lin}(x,y)$ defined by linear interpolation based on the values of u at the vertices of a triangle T defined by the vertices v_1, v_2 and v_3 .

Let h_i be the length of the edge connecting v_i and v_{i+1} where $v_4 = v_1$ and define the vectors $\underline{\hat{x}}$, $\underline{\hat{y}}$, and $\underline{\hat{z}}$ by

$$v_2 = v_1 + \underline{\hat{x}}, \quad v_3 = v_2 + \underline{\hat{y}}, \quad v_1 = v_3 + \underline{\hat{z}} \tag{4.1}$$

and consequently

$$\underline{\hat{x}} + \underline{\hat{y}} + \underline{\hat{z}} \ = \ 0. \tag{4.2}$$

where

$$\underline{\hat{x}} = \begin{bmatrix} \hat{x}_0 \\ \hat{x}_1 \end{bmatrix}, \ \underline{\hat{y}} = \begin{bmatrix} \hat{y}_0 \\ \hat{y}_1 \end{bmatrix}, \ \underline{\hat{z}} = \begin{bmatrix} \hat{z}_0 \\ \hat{z}_1 \end{bmatrix}$$

Define a reference trangle, T_{ref}, by the three nodal points:

$$v_1 = (0,0)^T \ , \ v_2 = (1,0)^T \ , \ v_3 = (0,1)^T \ , \tag{4.3}$$

Then the mapping from the triangle, T_{ref}, to the triangle, T is given by, Nadler [10]:

$$\underline{x} = v_1 \ + \ B\ \underline{\tilde{x}} \tag{4.4}$$

where $B \ = \ [\underline{\hat{x}}, -\underline{\hat{z}}]$, $\underline{\tilde{x}}$ is in the reference triangle . T_{ref} and $\underline{x}$ is the equivalent point in the original triangle T . The function u may then be expressed as

$$u(x,y) \ = \ \frac{1}{2} v_1^T\ H\ v_1 + \frac{1}{2} v_1^T\ H\ \underline{\tilde{x}} + \frac{1}{2} \underline{\tilde{x}}^T\ H\ v_1 + \frac{1}{2} \underline{\tilde{x}}^T\ B^T\ H\ B\ \underline{\tilde{x}} \quad where \ \underline{\tilde{x}} = \begin{bmatrix} x \\ y \end{bmatrix} \tag{4.5}$$

is defined on T_{ref}. Ignoring the constant and linear terms and expanding out using equation (4.4) gives

$$u(x,y) \ = \ \frac{1}{2}\ [(\underline{\hat{x}}^T H \underline{\hat{x}})x^2 + (-\underline{\hat{x}}^T H \underline{\hat{z}}) 2xy + (\underline{\hat{z}}^T H \underline{\hat{z}}) y^2\ ]\ , \ (x,y) \in T_{ref}. \tag{4.6}$$

Interpolating this by a linear function $u_{lin}(x,y)$ defined on T_{ref} by the nodal solution values gives

$$u_{lin}(x,y) \ = \ \frac{1}{2} \ \left[(\underline{\hat{x}}^T H \underline{\hat{x}})x + (\underline{\hat{z}}^T H \underline{\hat{z}}) y\ \right], \ (x,y) \in T_{ref} \tag{4.7}$$

and hence the linear interpolation error may be defined as as:

$$e_{lin}(x,y) = u_{lin}(x,y) \; - \; u(x,y) \tag{4.8}$$

and written in terms of variables defined on T_{ref} as

$$e_{lin}(x,y) = \frac{1}{2}\,[(\hat{\underline{x}}^T H \hat{\underline{x}})(x - x^2) \; - \; (-\hat{\underline{x}}^T H \hat{\underline{z}})2xy \; + \; (\hat{\underline{z}}^T H \hat{\underline{z}})(y - y^2)\,] \; . \tag{4.9}$$

This in turn may be written as

$$e_{lin}(x,y) = \frac{1}{2}\,\underline{W}^T\,\hat{\underline{d}},\;(x,y) \in T_{ref} \tag{4.10}$$

where

$$\underline{W}^T = \left[x - x^2,\; -2xy,\; y - y^2\right] \quad and \quad \hat{\underline{d}}^T \;= \left[\hat{\underline{x}}^T H \hat{\underline{x}}, -\hat{\underline{x}}^T H \hat{\underline{z}}, \hat{\underline{z}}^T H \hat{\underline{z}}\,\right] \quad .$$

4.1. Approximation in the H^1 Norm. The H^1 error norm is denoted by $||e_{lin}(x,y)||_{H^1}$ where

$$||e_{lin}(x,y)||^2_{H^1} = \int_T (e_{lin}(x,y))^2 + (e_{lin,x}(x,y))^2 + (e_{lin,y}(x,y))^2 dxdy \tag{4.11}$$

The integral of the first term on the righthand side is just equation (2.3) and so it remains to estimate the two remaining partial derivative terms, $e_{lin,x}$ and $e_{lin,y}$ with respect to x and y , which are expressed in terms of the partial derivatives $e_{lin,\tilde{x}}$ and $e_{lin,\tilde{y}}$ on the reference triangle by

$$\begin{bmatrix} e_{lin,x} \\ e_{lin,y} \end{bmatrix} \; = \; \left[B^{-1}\right]^T \begin{bmatrix} e_{lin,\tilde{x}} \\ e_{lin,\tilde{y}} \end{bmatrix} \tag{4.12}$$

where B is the matrix defined in equation (4.4) and where

$$\begin{bmatrix} e_{lin,\tilde{x}} \\ e_{lin,\tilde{y}} \end{bmatrix} = D\;\hat{\underline{d}}. \tag{4.13}$$

in which the matrix D is defined by

$$D \;=\; \frac{1}{2}\begin{bmatrix} 1-2x & -2y & 0 \\ 0 & -2x & 1-2y \end{bmatrix}$$

Hence from equations (4.10) , (4.11) and (4.12)

$$e_{lin,x}(x,y))^2 \; + \; (e_{lin,y}(x,y))^2 \; = \; \hat{\underline{d}}^T\; D^T\; \left[B^{-1}\right]\; \left[B^{-1}\right]^T D\;\hat{\underline{d}}. \tag{4.14}$$

Define the matrix B^* by

$$B^* \;=\; \begin{bmatrix} b_1 & b_2 \\ b_2 & b_4 \end{bmatrix} \;=\; \left[B^{-1}\right]\;\left[B^{-1}\right]^T \tag{4.15}$$

In the case of B as in equation (4.4) let

$$B \;=\; \begin{bmatrix} \hat{x}_0 & -\hat{z}_0 \\ \hat{x}_1 & -\hat{z}_1 \end{bmatrix}$$

then

$$B^* \;=\; \frac{1}{(D_{et})^2}\begin{bmatrix} \hat{z}_1^2 + \hat{z}_0^2 & \hat{z}_1\hat{x}_1 + \hat{z}_0\hat{x}_0 \\ \hat{z}_1\hat{x}_1 + \hat{z}_0\hat{x}_0 & \hat{x}_1^2 + \hat{x}_0^2 \end{bmatrix}$$

where $D_{et} \;=\; (-\hat{x}_0\hat{z}_1 + \hat{x}_1\hat{z}_0) = 2A$ where A is the area of the triangle. Hence from equation (4.10)

$$\int_T (e_{lin,x}(x,y))^2 + (e_{lin,y}(x,y))^2\; dxdy \;=\; 2A\int_{T_{ref}} \hat{\underline{d}}^T\; D^T B^* D\;\hat{\underline{d}}\; dxdy \tag{4.16}$$

where A is the area of the triangle and where
$D^T B^* D$ =

$$\frac{1}{4}\left[\begin{array}{ccc} (1-2x)^2 b_1 & (4x-2)(b_1 y + b_2 x) & (1-2x)(1-2y)b_2 \\ (4x-2)(b_1 y + b_2 x) & 4(b_1 y^2 + (b_1+b_2)xy + b_4 x^2) & (4y-2)(b_2 y + b_4 x) \\ (1-2y)(1-2x)b_2 & (4y-2)(b_2 y + b_4 x) & (1-2y)^2 b_4 \end{array}\right].$$

This may then be written as

$$(4.17) \qquad \int_T (\ e_{lin,x}(x,y))^2 \ + \ (e_{lin,y}(x,y))^2 \ dx \ dy \ = \ 2A \ \underline{\hat{d}}^T \ M \ \underline{\hat{d}}$$

where the components $[M]_{ij}$ of the matrix M are defined in terms of the integrals of the i,j th components of the matrix $D^T B^* D$ on the reference triangle. A straightforward but lengthy calculation gives

$$M \ = \ \frac{1}{24}\left[\begin{array}{ccc} b_1 & -b_1 & 0 \\ -b_1 & b_s & -b_4 \\ 0 & -b_4 & b_4 \end{array}\right].$$

where $b_s = 2(b_1 + b_2 + b_4)$ It is now necessary to express the vector $\underline{\tilde{d}}$ in terms of the vector of second directional derivatives along edges defined by

$$\underline{d}^T = \frac{1}{2} \ \left[\underline{\hat{x}}^T H \underline{\hat{x}}, \ \ \underline{\hat{y}}^T H \underline{\hat{y}}, \ \ \underline{\hat{z}}^T H \underline{\hat{z}}\right].$$

This is achieved by use of the vector identities defined by equation (4.2). For instance

$$\underline{\hat{y}}^T H \underline{\hat{y}} \ = \ (\underline{\hat{x}} + \underline{\hat{z}})^T \ H \ (\underline{\hat{x}} + \underline{\hat{z}})$$

and on expanding the righthand side of this we get

$$-\underline{\hat{x}}^T \ H \ \underline{\hat{z}} \ = \ \frac{1}{2}\underline{\hat{x}}^T \ H \ \underline{\hat{x}} + \frac{1}{2}\underline{\hat{z}} \ H \ \underline{\hat{z}} - \frac{1}{2}\underline{\hat{y}}^T \ H \ \underline{\hat{y}} \ .$$

From these identities and the definitions of the vectors $\underline{\tilde{d}}$ and $\underline{d}$ it follows that

$$\underline{\hat{d}} \ = \ N \ \underline{d}, \ \ where \ \ N = \left[\begin{array}{ccc} 2 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 2 \end{array}\right].$$

Using this to substitute for $\underline{\hat{d}}$ in equation (4.17) gives

$$(4.18) \qquad \int_T e_{lin,x}(x,y))^2 \ + \ (e_{lin,y}(x,y))^2 \ dx \ dy \ = \ 2 \ A \ \underline{d}^T \ N^T \ M \ N \ \underline{d} \ .$$

Define the matrix product $N^T M N$ by

$$N^T \ M \ N \ = \ \frac{1}{24}\left[\begin{array}{ccc} b_s & 2b_1 - b_s & 2b_2 \\ 2b_1 - b_s & b_s & 2b_4 - b_s \\ 2b_2 & 2b_4 - b_s & b_s \end{array}\right].$$

Further define

$$(4.19) \qquad \tilde{r}(\underline{d}) \ = \ 24 \ \underline{d}^T \ N^T \ M \ N \ \underline{d}$$

and expand out equation (4.18) in terms of the components of $\underline{d}$ which are the three directional derivatives along the edges to get:

$$(4.20) \qquad \tilde{r}(\underline{d}) \ = \ b_s(d_1 - d_2 + d_3)^2 + 4b_1(d_2 - d_3)d_1 + 4b_4(d_1 - d_2)d_3$$

This expression may be rewritten by noting that:

$$b_s \ = \ \frac{1}{(2A)^2} \ \left[||\underline{\hat{x}}||_2^2 + ||\underline{\hat{y}}||_2^2 + ||\underline{\hat{z}}||_2^2\right]$$

$$2b_1 - b_s = \frac{1}{(2A)^2}\left[-\|\underline{\hat{x}}\|_2^2 - \|\underline{\hat{y}}\|_2^2 + \|\underline{\hat{z}}\|_2^2 \right]$$

$$2b_4 - b_s = \frac{1}{(2A)^2}\left[\|\underline{\hat{x}}\|_2^2 - \|\underline{\hat{y}}\|_2^2 - \|\underline{\hat{z}}\|_2^2 \right]$$

$$2b_2 = \frac{1}{(2A)^2}\left[-\|\underline{\hat{x}}\|_2^2 + \|\underline{\hat{y}}\|_2^2 - \|\underline{\hat{z}}\|_2^2 \right]$$

where $\|\underline{\hat{x}}\|_2^2 = \underline{\hat{x}}^T\underline{\hat{x}}$ and similarly for the other terms. On combining these expressions in equation (4.20) we get

$$\tilde{r}(\underline{d}) = \frac{1}{4A^2}\left[\|\underline{\hat{x}}\|_2^2 D_1^2 + \|\underline{\hat{y}}\|_2^2 D_2^2 + \|\underline{\hat{z}}\|_2^2 D_3^2 \right] \tag{4.21}$$

where

$$D_1 = (-d_1 + d_2 + d_3)\ , \quad D_2 = (d_1 - d_2 + d_3) \ \ and\ D_3 = (d_1 + d_2 - d_3)$$

Hence on combining equations (2.3) (4.20) and (4.21)

$$\|e_{lin}(x,y)\|_{H_1}^2 = \frac{A}{12}\left[\frac{1}{15}\tilde{q}(\underline{d}) + \tilde{r}(\underline{d}) \right] \tag{4.22}$$

4.2. H1 norm Triangular Mesh Quality Indicator . The results in the previous section make it possible to define the mesh quality indicator in the same way as in Section 2 in that a scaled form of the error is used. Rewrite $\tilde{r}$ as

$$\tilde{r}(\underline{d}) = r_1(d) + r_2(d) + r_3(d) \tag{4.23}$$

where

$$r_1(d) = \frac{1}{4A^2}\|\underline{\hat{x}}\|_2^2 D_1^2, \quad r_2(d) = \frac{1}{4A^2}\|\underline{\hat{y}}\|_2^2 D_2^2, \quad r_3(d) = \frac{1}{4A^2}\|\underline{\hat{z}}\|_2^2 D_3^2,$$

Define

$$r_{max} = max(r_1(d),\ r_2(d),\ r_3(d))$$

In a similar way to as in Section 3 scale the L_2 part of the error by 0.8 d_{max}^2 and the term $\tilde{r}(\underline{d})$ by $3r_{max}$, thus giving two quantities that are one if $d_i = d_{max}$ and $r_i = r_{max}$. The weighted sum of these quantities is then obtained by then multiply the L_2 term by 0.8 d_{max}^2 and the remaining term by $3r_{max}$ and divide everything by the sum of these terms to get

$$\tilde{Q}_H(\tilde{\underline{d}}) = \left[(\frac{1}{15}q(\tilde{\underline{d}}) + r(\tilde{\underline{d}}) \right] / \left[0.8d_{max}^2 + 3r_{max} \right] \tag{4.24}$$

where $\tilde{\underline{d}} = [|d_1|, |d_2|, |d_3|]$.

5. Numerical Experiments.

5.1. Babuska and Aziz's Example. The first problem considered is the example used in the seminal paper of Babuska and Aziz [2] in which triangles of the form of that in Figure 2.1 are used to model a flow with a horizontal component $u_{xx} = 1$ and no other non-zero components $u_{xy} = 0$ and $u_{yy} = 0$. In the notation of Babuska and Aziz $H = \alpha h$ in figure 1 and the cases $\beta = 1$ and $\beta = \frac{1}{2}$ are considered. Hence $U(x,y) = \frac{1}{2}x^2$ and $U_{lin}(x,y) = \frac{1}{2}x + \beta(\beta-1)y/(2\alpha)$ and so

$$e_{lin,x}(x,y))^2 + (e_{lin,y}(x,y))^2 = (x - \frac{1}{2})^2 + (\beta(\beta-1)/(2\alpha)^2$$

thus showing a potential source of large errors for small values of α .
For a general value of β , $d_1 = h^2, d_2 = (1-\beta)^2h^2, d_3 = \beta^2\ h^2$,

$$\|\underline{\hat{x}}\|_2^2 = h^2\ , \|\underline{\hat{y}}\|_2^2 = h^2\ [\alpha^2 + (1-\beta)^2],\ \|\underline{\hat{z}}\|_2^2 = h^2\ [\alpha^2 + \beta^2]\ .$$

Hence

$$D_1 = h^2 2\beta(1-\beta), \quad D_2 = h^2 2\beta, \quad D_3 = h^2 2(1-\beta). \tag{5.1}$$

Hence from equation (2.3)

$$q(\underline{d}) = h^4 \left[(1+\beta^2+(1-\beta)^2)^2 + (1-\beta)^4 + 1 + \beta^4\right] \tag{5.2}$$

$$r(\underline{d}) = 4h^2 \left[\frac{3\beta^2(1-\beta)^2}{\alpha^2}) + (1-\beta)^2 + \beta^2\right] \tag{5.3}$$

In the case when $\beta = 1$

$$\int_T (e_{lin}(x,y))^2 dx\, dy = \frac{\alpha h^2}{180} 6h^4 \text{ and } q_{aniso} = \frac{1}{2} \tag{5.4}$$

thus that only two edges contribute to the error, whereas in the case when $\beta = \frac{1}{2}$

$$\int_T (e_{lin}(x,y))^2 dx\, dy = \frac{\alpha h^2}{180}\frac{54}{16} h^4 \text{ and } q_{aniso} = \frac{9}{32} \tag{5.5}$$

Thus showing that the error in the L_2 norm is slightly less when $\beta = \frac{1}{2}$ and from the anisotropy indicator value of 9/32 showing that one edge contributes significantly more to the error than the other two. In the case of the H^1 norm when $\beta = 1$,

$$\tilde{r}(\underline{d}) = 4h^2 , \quad r_{max} = 4h^2 ,$$

and

$$||e_{lin}(x,y)||^2_{H^1} = \frac{\alpha h^2}{12}\left[\frac{12}{15}h^4 + 4h^2\right] \text{ and } Q_H = \frac{60+6h^2}{180+12h^2)} \tag{5.6}$$

Hence as $h \downarrow 0$ the value of the indicator tends to 1/3, thus showing that one edge is primarily responsible for the error.

In contrast when $\beta = \frac{1}{2}$

$$\tilde{r}(\underline{d}) = 4h^2\left[\frac{3}{16\alpha^2} + \frac{1}{2}\right] , \quad r_{max} = 4h^2\left[\frac{1}{16\alpha^2} + \frac{1}{4}\right]$$

$$||e_{lin}(x,y)||^2_{H^1} = \frac{\alpha h^2}{12}\left[\frac{54}{240}h^4 + 4h^2(\frac{3}{16\alpha^2} + \frac{1}{2})\right] \tag{5.7}$$

and

$$Q_H = \frac{\frac{9}{40}h^2 + (\frac{3}{4\alpha^2} + 2)}{\frac{12}{15}h^2 + (\frac{3}{4\alpha^2} + 3)} \tag{5.8}$$

Hence as $\alpha \downarrow 0$ the value of the indicator tends to 1 thus showing that all three edges contribute to the error. These results are interesting because they show that in the L_2 norm $\beta = \frac{1}{2}$ is more accurate whereas in the H^1 norm for $\alpha < 0.4629$, $\beta = 1$ is more accurate. In particular in the case when

$$\alpha = h^p, p > 0 \tag{5.9}$$

as $h \downarrow 0$ then if $0 < \beta < 1$ the error in the H^1 norm is h^{1-p} and h^1 for $\beta = 1$ or $\beta = 0$. This indicates a considerably worse situation than that indicated by Babuska and Aziz who considered the same example but with $\alpha = h^p$ with $p > 5$.

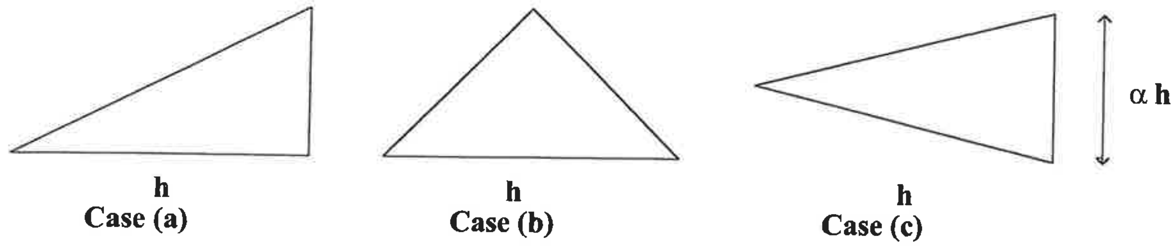

FIG. 5.1. *Boundary Layer Flow Example Triangles.*

TABLE 5.1
Mesh Quality Indicator Values L_2 Norm

α	Case (a)	Case (b)	Case (c)	Error Ratio a/c	Error Ratio b/c
1.0	0.49	0.49	0.29	1.8	1.70
0.1	0.42	0.42	0.35	1.8	1.40
0.038	0.35	0.34	0.53	1.7	1.00
0.02	0.30	0.29	1.00	1.5	0.71
0.01	0.28	0.30	0.68	1.3	0.44
0.001	0.42	0.29	0.50	1.0	0.47
0.0001	0.49	0.28	0.50	1.0	0.55

5.2. Boundary Layer Flow Example. The performance of this indicator may be illustrated by considering anisotropic flow, such as that in a viscous boundary layer, in which the three triangles defined as Case(a), Case(b) and Case(c) in Figure 5.1 are used to model a flow with a weak horizontal component $u_{xx} = 1$ an intermediate cross derivative $u_{xy} = 100$ and and a strong vertical component $u_{yy} = 10000$. Case(a) is representative of a triangle thought to be especially suitable for such flows while Case(b) is closer to the type of triangles produced by unstructured mesh generators. Berzins [3] obtained the results in Table 5.1 which show the values of q_{aniso} for the three triangles as the height of the triangles α is varied. Also shown is the ratio of the L_2 errors for Case (a) and Case (b) divided by the error in Case(c). The table shows that in the case when $\alpha < 0.04$ triangles such as that in Case(c) are best. These results are explained by the Indicator values and the values of d_{max} which are $(1 + 100\alpha)^2$ $(0.5 + 100\alpha)^2$ and $(1 + 50\alpha)^2$ for cases (a) (b) and (c) respectively. For very small values of α anisotropy is not a key factor as the effective dominant flow direction is the horizontal one.

In the case of the L_2 norm triangles of the type in Case (b) give the smallest error for small values of α. Table 5.2. shows the corresponding results in the H^1 norm , in which case however such Case (b) triangles are the worst of the three. In this case the mesh quality indicator is 1 indicating that all edges need to be refined. In contrast the mesh quality indicators for cases (a) and (c) have values close to 0.5 thus indicating that the error is distributed along two edges. This example shows that the triangle which appears to be best in the L_2 norm is by far the worst in the H^1 norm for small values of α.

6. Extensions to Tetrahedra. The above method for deriving a mesh quality indicator extends naturally to tetrahedra. The approach closely follows that defined above but with the obvious extensions described by Berzins [3] In this case the matrix

TABLE 5.2
Mesh Quality Indicator Values H^1 Norm

α	Case (a)	Case (b)	Case (c)	Error Ratio a/c	Error Ratio b/c
1.0	0.34	0.34	0.65	1.5	1.50
0.1	0.34	0.34	0.67	2.4	2.00
0.038	0.35	0.35	0.67	3.3	2.10
0.02	0.37	0.39	0.67	5.1	2.00
0.01	0.42	0.70	0.67	10.	2.40
0.001	0.61	0.69	0.68	430	3700
0.0001	0.66	0.96	0.50	10^4	10^8

$B*$ has the form

$$B^* = \begin{bmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{bmatrix}.$$

It is not however possible, in general to compute the entries of B^* in closed form, though with extended manipulation it may be possible to derive expressions similar to those above. The matrix D is now has the form

$$D = \begin{bmatrix} 1-2x & -2y & 0 & -2z & 0 & 0 \\ 0 & -2x & 1-2y & 0 & -2z & 0 \\ 0 & 0 & 0 & -2x & -2y & 1-2z \end{bmatrix}.$$

The remaning matrices etc are given by Berzins [3].

7. Extensions to Non-Quadratic Functions. The extension to te case of non-quadratic functions may be considered by assuming that the exact solution is locally quadratic. Bank [4] uses such an approach for example inside PLTMG and calculates estimates of second derivatives . Adjerid, Babuska and Flaherty [1] use a similar approach based on derivative jumps across edges to estimate the error. An alternative is approach is to use the ideas of Hlavacek et al. [8] to estimate nodal derivatives and hence second derivatives.

8. Conclusions. The H^1 norm-based mesh quality indicator discussed here allows the quality of the mesh to be assessed in a away that combines solution and geometry properties. The numerical examples considered here show that the choice of norm can be critical when assessing the mesh and reinforce the view that mesh quality cannot be considered independently of the solution error or the norm used to measure it.

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