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A Generalized Chebyshev Method for Non-linear Parabolic Equations in One Space Variable

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The derivation and implementation of a generalized Chebyshev method is described for the numerical solution of non-linear parabolic equations in one space dimension. The solution is obtained by using the method of lines and is approximated in the space variable by piecewise Chebyshev polynomial expansions. These expansions are normally few in number and of high order. It is shown that the method can be derived from a perturbed form of the original equation. A numerical example is given to illustrate its performance compared with the finite element and finite difference method.

A comparison of various Chebyshev methods is made by applying them to two-point eigenproblems. It is shown by analysis and numerical examples that the approach used to derive the generalized Chebyshev method is comparable, in terms of the accuracy obtained, with existing Chebyshev methods.

1. Introduction

ONE OF THE AUTHORS, Dew (1978), described a method for computing the numerical solution of quasi-linear parabolic partial differential equations in one space variable based on approximating the solution by a Chebyshev polynomial series expansion. It is well known that the solution to many parabolic problems cannot be adequately represented by a single polynomial and hence piecewise polynomials are used in practice, e.g. the finite element method and the collocation method (Madsen & Sincovec, 1978). Recently the authors (Berzins & Dew, 1980) extended and improved the Dew (1978) method and introduced a Chebyshev method based on piecewise Chebyshev series expansions. The purpose of this paper is to generalize the Chebyshev method so that it can be used to solve a wide class of non-linear parabolic equations in one space variable.

The generalized Chebyshev method is similar to the "global element" method described in Delves & Hall (1979) for elliptic PDEs, in the sense that we wish to split the spatial interval into a number of subintervals and within each subinterval construct a rapidly convergent global polynomial expansion. However, Delves & Hall derive their method from a functional embodying the boundary conditions whereas we derive our method directly from Chebyshev series expansions.

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For the sake of definiteness we shall consider equations of the type

$$c(x,t,u)\frac{\partial u}{\partial t}=\frac{1}{x^m}\frac{\partial}{\partial x}\left\{x^m\phi(x,t,u)\frac{\partial u}{\partial x}\right\}+f\left(x,t,u,\frac{\partial u}{\partial x}\right)$$
$$(x,t)\in[a,b]\times(t_s,t_e] \quad (1.1)$$

where

$$m=0,1,2,3, \qquad t_s<t_e$$

and

$$a<b, \qquad a\geqslant 0 \quad \text{when } m>0.$$

We assume the stability condition

$$0<\rho\leqslant\phi(x,t,u)\leqslant \mathrm{P}, \qquad \rho\leqslant c(x,t,u)\leqslant \mathrm{P}$$
$$\forall\ (x,t)\in[a,b]\times(t_s,t_e].$$

The functions c, ϕ and f may be discontinuous at a number of known points in (a,b) providing that these points are independent of t and the continuity of u and $\phi\ \partial u/\partial x$ is preserved.

The boundary conditions are taken to be

$$\left.\begin{aligned}\alpha(t)u+\beta(t)\frac{\partial u}{\partial x}&=g_a(t,u) \quad \text{at } x=a\\ \gamma(t)u+\delta(t)\frac{\partial u}{\partial x}&=g_b(t,u) \quad \text{at } x=b\end{aligned}\right\} t\in(t_s,t_e] \quad (1.2)$$

together with an initial condition

$$u(x,t)=K(x) \quad \text{at } t=t_s, \quad x\in[a,b]. \quad (1.3)$$

When $a=0$ and $m>0$ we restrict the left-hand boundary condition to the symmetry condition

$$\frac{\partial u}{\partial x}=0 \quad \text{at } x=a=0, \quad m>0. \quad (1.4)$$

2. Derivation of the Generalized Chebyshev Method

Let

$$a=x_0<x_1<\ldots<x_J=b$$

and partition the interval $[a,b]$ into the elements

$$\{I_j\colon\ [x_{j-1},x_j],\quad j=1,2,\ldots,J\}.$$

We can write the solution to (1.1), (1.2), (1.3) as

$$u(x,t)=\sum_{j=1}^{J}u_j(x,t) \quad (2.1)$$

where

$$u_j(x,t)\colon=u(x,t), \quad x\in I_j \text{ and zero elsewhere.}$$

Define the linear mapping W_j: $I_j \to [-1, 1]$. The basic idea of the method is to construct an approximate solution to $u(x, t)$ of the form

$$U(x, t) = \sum_{j=1}^{J} U_j(x, t) \tag{2.2}$$

where

$$U_j(x, t) = \sum_{i=0}^{N_j} a_{j,i} T_i(W_j(x)), \quad x \in I_j \text{ and zero elsewhere}$$

and $T_i(\,.\,)$ is a Chebyshev polynomial of degree i. The function U is chosen so that the continuity across each element is preserved and such that the boundary conditions (1.2), the initial condition and the interface condition

$$\phi(x_j, t, U_j)\frac{\partial}{\partial x} U_j = \phi(x_j, t, U_{j+1})\frac{\partial}{\partial x} U_{j+1}, \quad j = 1, 2, \ldots, J-1 \tag{2.3}$$

are all approximately satisfied.

To derive the method we approximate the differential equation (1.1) in the form

$$\frac{\partial}{\partial x} R_j(x, t) \approx Q_j(x, t), \quad x \in I_j, \quad j = 1, 2, \ldots, J \tag{2.4}$$

where

$$R_j(x, t) := \phi_j(x, t, U_j)\frac{\partial}{\partial x}\{U_j\}$$

and

$$Q_j(x, t) := c_j(x, t, U_j)\frac{\partial}{\partial t} U_j - f_j\left(x, t, U_j, \frac{\partial}{\partial x} U_j\right) - \frac{m}{x}\frac{\partial U_j}{\partial x}\phi_j.$$

The functions ϕ_j, c_j and f_j are defined in the same way as the function $u_j(x, t)$.

In the case when $a = 0$ and $m > 0$, the function Q_1 requires modification to handle the term

$$\frac{m}{x}\frac{\partial}{\partial x} U_1$$

correctly. It is easily shown at $x = a = 0$ that

$$\lim_{x \to 0}\left\{\frac{1}{x_m}\frac{\partial}{\partial x}\left(x^m \phi \frac{\partial u}{\partial x}\right)\right\} = (m+1)\frac{\partial}{\partial x}\left(\phi \frac{\partial u}{\partial x}\right)\bigg|_{x=0}$$

and hence at this point

$$Q_1 := \left\{c_1(x, t, U_1)\frac{\partial}{\partial t} U_1 - f_1(x, t, U_1)\right\}\Big/(m+1). \tag{2.5}$$

We write $\partial U_j/\partial x$ in the Chebyshev series expansion

$$\frac{\partial}{\partial x} U_j = \sum_{i=0}^{N_j} a_{j,i}^{(1)} T_i(W_j(x))$$

and approximate the functions R_j and Q_j by

$$\bar{R}_j = \sum_{i=0}^{N_j} r_{j,i} T_i(W_j(x))$$

and

$$\bar{Q}_j = \sum_{i=0}^{N_j} q_{j,i} T_i(W_j(x))$$

where the polynomials $\bar{R}_j$ and $\bar{Q}_j$ interpolate the functions R_j and Q_j, respectively, at the transformed Chebyshev points

$$X_j := \left\{ W_j(y_i):\ y_i = \cos\left(\frac{N_j - i}{N_j}\pi\right),\quad i = 0, 1, \ldots, N_j \right\}. \tag{2.6}$$

In the following derivation we shall make the approximation that $\partial \bar{R}_j/\partial x = \bar{Q}_j$ for each j; an analysis of the method is given in Section 4. We have the following well-known relationships between

(1) the coefficients $a_{j,i}$ and $a_{j,i}^{(1)}$

$$a_{j,N_j}^{(1)} = 0, \qquad a_{j,N_j-1}^{(1)} = 2N_j a_{j,N_j}/\mu_j$$

$$a_{j,i}^{(1)} = \frac{1}{\mu_j}\{2(i+1)a_{j,i+1}\} + a_{j,i+2}^{(1)}, \quad i = N_j - 2, N_j - 3, \ldots, 0 \tag{2.7}$$

and

(2) the coefficients $r_{j,i}$ and $q_{j,i}$

$$\frac{\mu_j}{2i}(q_{j,i-1} - q_{j,i+1}) = r_{j,i}, \quad i = 1, 2, \ldots, N_j - 1 \tag{2.8}$$

where

$$\mu_j := \tfrac{1}{2}(x_j - x_{j-1}), \quad j = 1, 2, \ldots, J$$

and the coefficient of $q_{j,0}$ should be doubled. These results can be found in Fox & Parker (1968).

Having computed the coefficients $a_{j,i}^{(1)}$ using (2.7) we can then compute the coefficients $r_{j,i}$ as shown in the next section.

To satisfy the boundary conditions (1.2) and the interface conditions (2.3) we extend the approach described in Berzins & Dew (1980). The main advantage compared with the Knibb & Scraton (1971, 1979) method is that it avoids the need to differentiate explicitly the boundary conditions w.r.t. t. From Equation (2.4) and the Chebyshev expansion for $\bar{Q}_j$ we have that

$$\bar{R}_j(x, t) = \sum_{i=0}^{N_j} q_{j,i} \int^{x} T_i(W_j(y))\,dy + C_1(t) \tag{2.9}$$

where the constant of integration is given by

$$C_1(t) = \frac{1}{2}\int_{-1}^{1} \left\{ \bar{R}_j(x, t) - \sum_{i=0}^{N_j} q_{j,i} e_i(W_j(x)) \right\} dx \tag{2.10}$$

and

$$e_i(x) = \int^x T_i(y)\,dy, \quad x \in [-1, 1].$$

Define

$$\bar{e}_i = \frac{1}{2}\int_{-1}^{1} T_i(y)\,dy \quad \text{and} \quad E_i = \frac{1}{2}\int_{-1}^{1} e(y)\,dy \tag{2.11}$$

then on substituting (2.10), (2.11) into (2.9) we have that

$$\bar{R}_j(x, t) = \sum_{i=0}^{N_j} r_{j,i}\bar{e}_i + \sum_{i=0}^{N_j} q_{j,i}\mu_j[e_i(W_j(x)) - E_i]. \tag{2.12}$$

Finally, replacing $\bar{R}_j$ by $\phi_j\,\partial U_j/\partial x$ and evaluating (2.12) at $x = x_{j-1}$ and $x = x_j$ we can estimate the derivative at each end of an element. An alternative expression for the boundary conditions (1.2) is then given by

$$\begin{aligned}
&\beta(t)\sum_{i=0}^{N_j} q_{1,i}\mu_1(e_i(-1) - E_i) \\
&\qquad = [g_a(t, U_1) - \alpha(t)U_1(a, t)]\phi(a, t, U_1) - \beta(t)\sum_{i=0}^{N_1} r_{1,i}\bar{e}_i \\
&\delta(t)\sum_{i=0}^{N_J} q_{J,1}\mu_J(e_i(1) - E_i) \\
&\qquad = [g_b(t, U_J) - \gamma(t)U_J(b, t)]\phi(b, t, U_J) - \delta(t)\sum_{i=0}^{N_J} r_{J,i}\bar{e}_i
\end{aligned} \tag{2.13}$$

and similarly at the interface of each element we have that

$$\begin{aligned}
&-\sum_{i=0}^{N_{j-1}} q_{j-1,i}\mu_{j-1}(e_i(1) - E_i) + \sum_{i=0}^{N_j} q_{j,i}\mu_j(e_i(-1) - E_i) \\
&\qquad = \sum_{i=0}^{N_{j-1}} r_{j-1,i}\bar{e}_i - \sum_{i=0}^{N_j} r_{j,i}\bar{e}_i, \quad j = 2, 3, \ldots, J.
\end{aligned} \tag{2.14}$$

The use of these formulas means that the function U satisfies an approximate form of the derivative boundary condition and derivative interface condition. However, in order that the method is stable, we must preserve the continuity of U across each element interface and satisfy exactly any Dirichlet boundary conditions.

When the function ϕ is identically equal to unity we can rewrite (2.12)

$$\left.\frac{\partial u}{\partial x}\right|_{x=x_k} \approx \sum_{i=0}^{N_j} [q_{j,i}\mu_j(e_i(W_j(x_k)) - E_i) + \tfrac{1}{2}a_{j,i}(1 - (-1)^i)/\mu_j]$$

where $k = j - 1$ or j. This is the expression for the derivative used in Berzins & Dew (1980).

3. Implementation Details

We shall describe the implementation of the method assuming that we have an ordinary differential equation solver which will compute the numerical solution to a general system of implicit ordinary differential equations of the form

$$\mathbf{F}(t, \mathbf{U}(t), d\mathbf{U}/dt) = 0. \tag{3.1}$$

Any ODE solver based on an implicit method can be extended to solve implicit ODEs of this form. For example Dew & Walsh (1981) describe how to extend Gear's method. In our case the vector $\mathbf{U}$ is ordered as

$$\mathbf{U}(t) = \begin{bmatrix} \mathbf{U}_1(t) \\ \mathbf{U}_2(t) \\ \vdots \\ \mathbf{U}_J(t) \end{bmatrix}, \quad \mathbf{U}_j(t) = \begin{bmatrix} U_{j,0} \\ U_{j,1} \\ \vdots \\ U_{j,N_j-1} \end{bmatrix}, \quad j = 1, 2, \ldots, J-1, \quad \mathbf{U}_J = \begin{bmatrix} U_{J,0} \\ U_{J,1} \\ \vdots \\ U_{J,N_J} \end{bmatrix}$$

where (see (2.6))

$$U_{j,i} := U(X_{j,i}, t), \qquad X_{j,i} := W_j(y_i), \quad \begin{array}{l} i = 0, 1, \ldots, N_j \\ j = 1, 2, \ldots, J. \end{array}$$

From the continuity condition (2.3) it follows that $U_{j,N_j} = U_{j+1,0}, j = 1, 2, \ldots, J-1$. The vector $\mathbf{F}$ is ordered in an equivalent manner.

Initially we must compute and save three sets of matrices. The matrix Ω_j defined by (see Fox & Parker, 1968)

$$[\Omega_j]_{i,k} := \frac{2}{N_j} T_i(y_k) \quad \text{(the first and last rows and columns should be halved)}$$

is used to map the solution vector into its Chebyshev coefficients, e.g.

$$\mathbf{a} = \begin{bmatrix} a_{j,0} \\ a_{j,1} \\ \vdots \\ a_{j,N_j} \end{bmatrix} = \Omega_j \begin{bmatrix} U_{j,0} \\ U_{j,1} \\ \vdots \\ U_{j,N_j} \end{bmatrix}.$$

(The reader should note that Ω_j is the inverse of the mapping matrix used in earlier papers.)

The second matrix, D_j, is used to estimate the values of $\partial U/\partial x$ at the integration points $X_{j,i}$. We first define

$$[\bar{D}_j]_{i,k} := \frac{d}{dx} T_i(X_{j,k}), \quad i, k = 0, 1, \ldots, N_j$$

then

$$D_j := \bar{D}_j \Omega_j.$$

The matrix $\bar{D}_j$ can readily be constructed using (2.7). Finally, we define a matrix $C_j = \bar{C}_j \Omega_j$ where

$$[\bar{C}_j]_{0,k} = e_k(-1) - E_k, \qquad [\bar{C}_j]_{N_j,k} = -e_k(1) + E_k, \quad k = 0, 1, \ldots, N_j,$$

$$[\bar{C}_j]_{i+1,i} = 0{\cdot}5/(i+1), \qquad [\bar{C}_j]_{i+1,i+2} = -0{\cdot}5/(i+1); \quad i = 1, \ldots, N_j - 2,$$

$$[\bar{C}_j]_{1,0} = 1{\cdot}0, \qquad [\bar{C}_j]_{1,2} = -0{\cdot}5 \quad \text{and} \quad [C_j]_{i,k} = 0 \text{ otherwise}$$

where the top and bottom rows of $\bar{C}_j$ arise from the left-hand side of (2.13) and the middle rows from the left-hand side of (2.8).

The matrices Ω_j, D_j and C_j depend only on N_j and not on the size or position of the element. Therefore we need only save one copy of each matrix when N_j is constant for each element. The following implementation details illustrate how straightforward it is to program the method. Although, for simplicity, we have only considered one parabolic equation the method extends naturally to systems.

Evaluation of the Function $\mathbf{F}(t, \mathbf{U}, d\mathbf{U}/dt)$

Given the values of t, $\mathbf{U}$, $d\mathbf{U}/dt$ we require to evaluate the vector $\mathbf{F}$. This can be formed element by element and we therefore describe the implementation details for the jth element where $j = 1, 2, \ldots, J$.

To form the part of the vector $\mathbf{F}$ corresponding to the points $\{X_{j,i}\colon\ i = 0, 1, \ldots, N_j\}$ in the jth element, we first copy the solution values $\mathbf{U}$ at these points into a temporary vector $\mathbf{U}^{(T)}$† and use it to estimate the derivative, $\partial u/\partial x$ at the points using the relationship

$$\dot{\mathbf{U}}_j = \mu_j^{-1} D\mathbf{U}^{(T)}.$$

From the definition of the PDE functions, ϕ, c, f we can form the vector $\mathbf{Q}_j$ whose components are given by

$$Q_{j,i} = c_{j,i}\frac{dU_{j,i}}{dt} - f_{j,i} - \frac{m}{X_{j,i}}\dot{U}_{j,i}\phi_{j,i}, \quad i = 0, 1, \ldots, N_j$$

where

$$\phi_{j,i} := \phi(t, X_{j,i}, U_i^{(T)})$$

and $c_{j,i}$, $f_{j,i}$ are defined in the same way. When $X_{j,i} = 0$ and $m > 0$ we must use the modification (2.5). To compute the coefficients $\mathbf{r}_j$ we use the mapping matrix Ω_j,

$$\mathbf{r}_j = \Omega_j \mathbf{R}_j \quad \text{where } R_{j,i} := \phi_{j,i}\dot{U}_{j,i}.$$

We can now form the components of $\mathbf{F}$ corresponding to the internal points of the jth element, i.e.

$$F_{j,i} = Q_i^{(T)} - r_{j,i}, \quad i = 1, 2, \ldots, N_j - 1$$

where the temporary vector $\mathbf{Q}^{(T)}$ is defined by

$$\mathbf{Q}^{(T)} = \mu_j C_j \mathbf{Q}_j.$$

Finally, we need to compute the components of $\mathbf{F}$ that correspond to the left- and right-hand end points of the jth element. We first form the sum

$$r^{(s)} := \sum_{i=0}^{N_j} r_{j,i} e_i$$

where e_i is defined by (2.11).

† We have introduced the vector $\mathbf{U}^{(T)}$ to simplify the description, in a computer program it is only necessary to reference the required elements of $\mathbf{U}$.

For the first element, $j = 1$, the component of $\mathbf{F}$ corresponding to the left-hand boundary condition (at $x = a$) is given by

$$F_{1,0} = \beta(t)(Q_0^{(E)} + r^{(s)}) + (g_a(t, U_{1,0}) - \alpha(t)U_{1,0})\phi_{1,0}.$$

Otherwise we form the component corresponding to the derivative interface condition

$$F_{j,0} = (Q_0^{(E)} + r^{(s)}) + r_{j-1}^{(d)}$$

where the estimate of $-\phi_{j,N_j}(\partial U_j/\partial x)$ at the point X_{j,N_j} is given by

$$r_j^{(d)} := Q_{N_j}^{(E)} + r^{(s)}.$$

For the last element, $j = J$, we must form the component of $\mathbf{F}$ which corresponds to the right-hand boundary condition (at $x = b$). This component is given by

$$F_{J,N_J} = -r_j^{(d)}\delta(t) + (g_b(t, U_{J,N_J}) - \gamma(t)U_{J,N_J})\phi_{J,N_J}.$$

This completes the implementation details for forming the vector $\mathbf{F}$.

To compute a numerical solution to the system of ODEs, $\mathbf{F} = 0$, we must supply to the ODE solver:

(i) a routine to evaluate the vector $\mathbf{F}$ (as above),
(ii) a routine to evaluate the Jacobian matrix for $\mathbf{F}$ and
(iii) the initial values of $\mathbf{U}(t_s)$ and $d\mathbf{U}(t_s)/dt$.

The Jacobian matrix of $\mathbf{F}$ has the form

$$\hat{J} := \frac{1}{\sigma} A(\mathbf{U}, t) + \partial\mathbf{F}/\partial\mathbf{U}$$

where σ is a parameter which involves the timestep and is supplied by the ODE solver. The matrix $\partial\mathbf{F}/\partial\mathbf{U}$ can be computed by numerical differencing. Providing that the function $\partial U/\partial t$ is continuous for $x \in (a, b)$ at each $t \in (t_s, t_e]$ the matrix A is given by:

Let

$$\begin{aligned} K_j &= 0 && \text{for } j = 1 \\ &= \sum_{s=1}^{j-1} N_s && \text{for } j \neq 1 \end{aligned}$$

then

$$A_{K_j+i, K_j+k} := \mu_j C_{i,k} c_{j,k}, \quad k, i = 0, 1, 2, \ldots, N_j$$

unless $j \neq 1$, $i = k = 0$ in which case

$$A_{K_j, K_j} = A_{K_j, K_j} + \mu_j C_{0,0} c_{j,0}$$

for $j = 1, 2, \ldots, J$.

All other elements of A are zero and the first and last rows must be modified to correspond to the boundary conditions

$$A^{(1)}_{1,k} = \beta A^{(1)}_{1,k}, \qquad k = 0, 1, \ldots, N_1$$

and (3.2)

$$A^{(J)}_{N,K_J+k} = -\delta A_{N,K_J+k}, \quad k = 0, 1, \ldots, N_J$$

where

$$N = \sum_{j=1}^{J} N_j + 1.$$

The general structure of the Jacobian matrix $\hat{J}$ is best illustrated by an example. In the case when $J = 3$ (three elements) and $N_j = 2$

$$\hat{J} = \begin{bmatrix} \times & \times & \times & & & & \\ \times & \times & \times & & & & \\ \times & \times & \times & \times & \times & & \\ & & \times & \times & \times & & \\ & & \times & \times & \times & \times & \times \\ & & & & \times & \times & \times \\ & & & & \times & \times & \times \end{bmatrix}$$

where the crosses denote the non-zero elements.

The initial values $\mathbf{U}(t_s)$ can be determined from the initial condition (1.3). To compute the values of $d\mathbf{U}/dt$ at $t = t_s$ we write the vector $\mathbf{F}$ in the form

$$A\frac{d\mathbf{U}}{dt} = \hat{\mathbf{F}}(t, \mathbf{U}(t))$$

and put $t = t_s$. We can then solve for the values of $d\mathbf{U}/dt$ providing that the matrix A is non-singular. Clearly from (3.2) when either $\beta = 0$ or $\delta = 0$ the matrix A is singular. To avoid this difficulty we modify the equation at the boundary. For example, suppose that $\beta = 0$ and $\delta \neq 0$ then at $x = a$

$$\alpha(t)U_1(a, t) - g_a(t, U_1(a, t)) = 0 \tag{3.3}$$

From equation (2.12) we have that

$$\sum_{i=0}^{N_1} r_{1,i}(-1)^i = \sum_{i=0}^{N_1} q_{1,i}\mu_1(e_i(-1) - E_i) - \sum_{i=0}^{N_1} r_{1,i}\bar{e}_i \tag{3.4}$$

and on multiplying (3.3) by ϕ_1 at $x = a$ and adding (3.4) we get the condition

$$\sum_{i=0}^{N_1} q_{1,i}\mu_1(e_i(-1) - E_i) = (g_a(t, U_1) - \alpha(t)U_1(a, t))\phi(a, t, U_1) - \sum_{i=0}^{N_1} r_{1,i}(\mathbf{e}_i - (-1)^i). \tag{3.5}$$

The first row of A and the first element of $\hat{\mathbf{F}}$ can be modified in an obvious manner and since the modified A is now non-singular we can compute an estimate of

$d\mathbf{U}(t_s)/dt$. Use of (3.5) however means that

$$\alpha(t)U_1(a, t) \approx g_a(t, U_1(a, t))$$

and we cannot use (3.5) in the time integration as the method would be unstable.

4. The Perturbed Differential Equation

A feature of all Chebyshev methods is that it is easy to determine the perturbed differential equation which the method exactly satisfies. We are only concerned here with the approximation in the spatial variable x. Knowing the perturbed differential equation enables us to compare the method with existing Chebyshev methods.

From the derivation of the global elements method given in Section 2 we have that the functions Q_j and R_j are evaluated at the points $x \in X_j$ for $j = 1, 2, \ldots, J$ and then mapped into their Chebyshev coefficients using the mapping matrix Ω_j defined in Section 3. Thus we have replaced the functions Q_j and R_j by the polynomials $\bar{Q}_j$ and $\bar{R}_j$ which interpolate the functions at the points $x \in X_j$ and are zero outside the interval I_j. We define the interpolation errors as

$$E^{(R)} := \sum_{j=1}^{J} E_j^{(R)}, \qquad E_j^{(R)} := \phi_j \frac{\partial U_j}{\partial x} - \bar{R}_j$$

and

$$E^{(Q)} := \sum_{j=1}^{J} (Q_j - \bar{Q}_j). \tag{4.1}$$

LEMMA 1. *If the coefficients $r_{j,i}$ and $q_{j,i}$ satisfy the relationships (2.8) then*

$$\phi_j \frac{\partial U_j}{\partial x} - E_j^{(R)} - P_j^{(G)}(W_j(x), t) = \int^x \bar{Q}_j \, dx + C_1(t) \tag{4.2}$$

where

$C_1(t)$ is a constant of integration and

$$P_j^{(G)}(y, t) = r_{j,N_j} T_{N_j}(y) - \frac{\mu_j}{2N_j} q_{j,N_j-1} T_{N_j}(y) - \frac{\mu_j}{2(N_j+1)} q_{j,N_j} T_{N_j+1}(y), \quad y \in [-1, 1] \tag{4.3}$$

$$= 0 \quad \textit{elsewhere}.$$

Proof. The proof follows by expressing the polynomials $\bar{Q}_j$ and $\bar{R}_j$ in their Chebyshev series expansions (2.5), integrating and equating the coefficients of the Chebyshev polynomials for each element.

From equation (4.2) we can deduce that

$$\phi_j \frac{\partial U_j}{\partial x} = \int^x \bar{Q}_j(z, t) \, dz + \frac{1}{2\mu_j} \int_{x_{j-1}}^{x_j} (\bar{R}_j(x, t) - \int^x \bar{Q}_j(z, t) \, dz) \, dx + E_j^{(R)} + P_j^{(G)}(W_j(x), t) - \frac{1}{2} \int_{-1}^{1} P_j^{(G)}(y, t) \, dy, \quad x \in I_j \tag{4.4}$$

and hence in (2.13) and (2.14) we have used a perturbed form of the derivative given by

$$\frac{\partial U_j}{\partial x} - \psi_j(W_j(x_k), t)/\phi_{j, x_k}, \quad k = j-1, j$$

where

$$\psi_j(y, t) := P_j^{(G)}(y, t) - \frac{1}{2}\int_{-1}^{1} P_j^{(G)}(y, t)\, dy$$

and

$$\phi_{j, x_k} := \phi_j(x_k, t, U_j), \quad x_k \in I_j. \blacksquare$$

We can summarize these results in the following theorem:

THEOREM 1. *Let the function $U(x, t)$ defined by (2.2) satisfy the system of ODEs*

$$\mathbf{F}\left(t, \mathbf{U}, \frac{d\mathbf{U}}{dt}\right) = 0$$

where the vector $\mathbf{F}$ *is defined by the algorithm given in Section 3. Then $U(x, t)$ is the exact solution of the differential equation*

$$\frac{1}{x^m}\frac{\partial}{\partial x}\left\{x^m\phi(x, t, U)\frac{\partial U}{\partial x}\right\} - \frac{\partial}{\partial x}(E^{(R)} - P^{(G)}) + f\left(x, t, U, \frac{\partial U}{\partial x}\right)$$
$$= c(x, t, U)\frac{\partial U}{\partial t} - E^{(Q)}, \quad (x, t) \in \bigcup_{j=1}^{J} (x_{j-1}, x_j)\mathrm{x}(t_s, t_e),$$

subject to the conditions

$$\alpha U_1 + \beta\frac{\partial U_1}{\partial x} = g_a(t, U_1) + \psi_1(-1, t)\beta/\phi_{1,a} \quad \text{at } x = a$$

$$\phi_{j, x_j}\frac{\partial U_j}{\partial x} = \phi_{j+1, x_j}\frac{\partial U_{j+1}}{\partial x} + \psi_j(1, t) - \psi_{j+1}(-1, t), \quad j = 1, \ldots, J-1$$

and

$$\gamma U_J + \delta\frac{\partial U_J}{\partial x} = g_b(t, U_J) + (\psi_J(1, t)\delta)/\phi_{J,b} \quad \text{at } x = b$$

where

$$P^{(G)}(x, t) = \sum_{j=1}^{J} P_j^{(G)}(W(x), t)$$

is the perturbation term.

5. Comparison with Existing Chebyshev Methods

In this section we compare the approach used to derive the generalized Chebyshev method with existing Chebyshev methods. For the non-polar problem, $m = 0$, with $\phi \equiv 1$ we can compare the method with the Berzins & Dew (1980) method. The main difference is that in the Berzins & Dew method we apply prior integration twice (see Fox & Parker (1968)) to obtain an extension of the Knibb & Scraton (1971) method. Whereas in the present algorithm we apply prior integration once and obtain the derivative of the solution directly from the Chebyshev series expansion of the solution.

We can easily modify the generalized Chebyshev method so that it is equivalent to the Berzins & Dew (1980) method by perturbing the relationship between the Chebyshev coefficients of U and $\partial U/\partial x$. That is we replace in Equation (2.7)

$$\left.\begin{aligned} &a_j, N_j \quad \text{by } a_{j,N_j} - q_{j,N}/[4(N_j+1)N_j] \\ \text{and}\quad &a_{j,N_j-1} \quad \text{by } a_{j,N_j-1} - q_{j,N_j-1}/[4N_j(N_j-1)]. \end{aligned}\right\} \tag{5.1}$$

The Simple Eigenvalue Problem

To illustrate the effect of applying prior integration once or twice we consider the simple eigenvalue problem

$$\left.\begin{aligned} &\frac{d^2v}{dx^2} + \lambda v = 0, \quad x \in [0, 1] \\ \text{subject to}\quad &\alpha v + \beta\frac{dv}{dx} = 0 \quad \text{at } x = 0 \\ \text{and}\quad &\gamma v + \delta\frac{dv}{dx} = 0 \quad \text{at } x = 1 \\ \text{where}\quad &\alpha(\gamma+\delta) - \gamma(\beta-\alpha) \neq 0. \end{aligned}\right\} \tag{5.2}$$

We approximate $v(x)$ by a polynomial of degree N, written as

$$v_N(x) = \sum_{i=0}^{N} a_i T_i(2x-1), \quad x \in [0, 1]$$

and obtain an estimate of the eigenvalues of (5.2) from the eigenvalues of the generalized matrix eigenvalue problem

$$A\mathbf{a} + \lambda^* B\mathbf{a} = 0 \tag{5.3}$$

where

$$\mathbf{a} = [a_0 \;\; a_1 \;\; a_2 \;\; \ldots \;\; a_N]^T.$$

The $(N+1)\times(N+1)$ matrices A and B are formed using prior integration either once or twice and by adding two extra equations which arise from the boundary conditions. We have two alternatives, either

(a) we make v_N exactly satisfy the boundary conditions, e.g.

$$\alpha \sum_{i=0}^{N} a_i(-1)^i - \beta \sum_{i=0}^{N} i^2(-1)^i a_i = 0,$$

as suggested in Knibb & Scraton (1971) or

(b) we can use an approximate formula for the derivative of v_N, so that v_N satisfies a perturbed form of the boundary conditions, as suggested in Berzins & Dew and used in this paper.

The following theorem compares the different methods. We shall use the $L_2[0, 1]$ norm defined by

$$\|u\|^2 = \int_0^1 f^2\, dx, \quad f \in L_2[0, 1].$$

THEOREM 2. *Let* $(\lambda^*, \mathbf{a}^*)$ *denote an eigenpair of* (4.7) *and define*

$$v^*(x) := \sum_{i=1}^{N} a_i^* T_i(2x-1)$$

where $\mathbf{a}^*$ *is normalized such that* $\|v^*\| = 1$. *Further let* (λ^*, v^*) *be the closest eigenpair to the kth eigenpair of* (4.6) *in the sense that* $|\lambda_k - \lambda^*|/|\lambda_k|$ *is a minimum, then*

$$\frac{|\lambda_k - \lambda^*|}{|\lambda_k|} \leqslant |\lambda^*| \max(|a_{N-1}|, |a_N|) O(N^{-P}),$$

where

$P = 2$ *for prior integration once or twice using alternative* (*b*) *for the boundary conditions,*
$P = 1$ *for prior integration once using alternative* (*a*) *for the boundary conditions,*
$P = 0$ *for prior integration twice using alternative* (*a*) *for the boundary conditions.*

Proof. The results for prior integration twice is obtained from the perturbation equation (see Berzins & Dew, 1980, and from the result of Theorem 2 in Basford & Dew, 1980). For prior integration once we have from the result of Theorem 1 that

$$\frac{d^2(v_N - P(x))}{dx^2} + \lambda^* v_N = 0$$

where

$$P(x) = \lambda^* \left\{ \frac{a_{N-1}}{4N} \left(\frac{T_{N+1}(y)}{(N+1)} - \frac{T_{N-1}(y)}{(N-1)} \right) + \frac{a_N}{4(N+1)} \left(\frac{T_{N+2}(y)}{(N+2)} - \frac{T_N(y)}{N} \right) + C_1 x + C_2 \right\},$$

$$y = 2x - 1,$$

and C_1, C_2 are chosen so that $P(x)$ satisfies the boundary conditions (5.2). The result then follows as for prior integration twice.

We might expect that there is little to choose between prior integration once or twice providing that we handle the boundary conditions as suggested in Berzins & Dew (1980). We see from the results in Table 1(i) that this is indeed the case for $\alpha = 0$, $\beta = 1$, $\gamma = 1$, $\delta = 0$ where

$$E(\lambda_i) = |\lambda_i - \lambda_i^*|/|\lambda_i|, \quad i = 1, 2, 3 \tag{5.4}$$

and λ_1, λ_2, λ_3 are the leading eigenvalues of (5.2).

TABLE 1

Errors in the eigenvalues for the polar operator

$$\frac{d^2v}{dx^2} + \lambda v = 0, \quad x \in [0, 1], \quad \frac{dv}{dx} = 0 \quad \text{at } x = 0 \text{ and } v = 0 \text{ at } x = 1$$

Errors ($N = 8$)	Exact boundary conditions (a)		Approx. boundary conditions (b)	
	(i)	(ii)	(i)	(ii)
$E(\lambda_1)$	0·105E–6	0·239E–7	0·263E–9	0·861E–10
$E(\lambda_2)$	0·245E–3	0·495E–4	0·809E–5	0·246E–5
$E(\lambda_3)$	0·369E–2	0·985E–3	0·232E–3	0·124E–3

(i) Prior integration twice, (ii) prior integration once.

The Polar Eigenvalue Problem

In deriving a generalized Chebyshev method for an equation with a polar operator we could incorporate the method suggested in Dew & Scraton (1972). To evaluate whether this would be worthwhile we compare the method given in this paper with the Dew & Scraton method applied to the polar eigenvalue problem

$$\frac{1}{x^m}\frac{d}{dx}\left(x^m \frac{dv}{dx}\right) + \lambda v = 0, \quad x \in [0, 1]$$

subject to (5.5)

$$\frac{dv}{dx} = 0 \quad \text{at } x = 0 \quad \text{and} \quad \gamma u + \delta \frac{dv}{dx} = 0 \quad \text{at } x = 1$$

where γ and δ cannot both be zero.

In the Dew & Scraton (1972) method the interval of integration was extended to $[-1, 1]$ and $v(x)$ was restricted to being an even function. To extend the method to piecewise Chebyshev expansions we must be able to map the interval $[0, 1]$ onto $[-1, 1]$.

We rewrite the differential equation into the form

$$\frac{d^2v}{dx^2} + \left(\frac{m}{x}\frac{dv}{dx} + \lambda v\right) = 0 \tag{5.6}$$

and again approximate $v(x)$ by $v_N(x)$. On applying prior integration once or twice we obtain

$$A\mathbf{a}+B(\mathbf{g}+\lambda\mathbf{a})=0 \tag{5.7}$$

where the matrices A and B are the same as the matrices occurring in (5.3) with $\alpha=0$, $\beta=1$,

$$G(x) := \frac{m}{x}\frac{dv_N}{dx} = \sum_{i=0}^{\infty} g_i T_i(2x-1) \tag{5.8}$$

and

$$\mathbf{g}=[g_0 \;\; g_1 \;\; \dots \;\; g_N]^T.$$

The boundary conditions can be handled using alternative (a) or (b).

In the method given in Section 2 we estimate the coefficients g_i directly from (5.8) whereas in the Dew & Scraton method we integrate (5.6) and replace $v(x)$ by $v_N(x)$ to give

$$G(x) = -\frac{m}{x}\left[\int^x (G(x)+\lambda v_N(x))\,dx + C\right] \tag{5.9}$$

where

$$C = \frac{v_N(1)-v_N(0)}{2} - \frac{1}{2}\int_0^1 \int^y \{G(y)+\lambda v_N(y)\}\,dy.$$

On writing (5.9) in the form

$$xG(x) = -m\int^x \{G(y)+\lambda v_N(x)\}\,dx + C$$

and on equating the coefficients of $T_1(2x-1)$, $T_2(2x-1)$, . . ., $T_{N+1}(2x-1)$ we find that

$$2ig_i+(i+m)g_{i-1}+(i-m)g_{i+1} = 4\lambda(a_{i-1}-a_{i+1}), \quad i=1,2,\dots,N+1. \tag{5.10}$$

By ignoring the coefficients g_{N+1}, g_{N+2}, the first $N+1$ equations of (5.10) can be written in the matrix form

$$M_1\mathbf{g} = \lambda M_2\mathbf{a} \Rightarrow \mathbf{g} = \lambda R\mathbf{a} \tag{5.11}$$

where M_1 and M_2 are upper tridiagonal matrices. The elements of the matrix R are independent of the parameter N. Equation (5.10) also ensures that the equation obtained by equating the coefficient of T_0 is satisfied. For a general region $[a, b]$ this is not the case and we must include the equation arising from the term T_0 and ignore the final equation at $i=N+1$.

We can substitute (5.11) into (5.7) to give

$$A\mathbf{a}+\lambda B(I+mR)\mathbf{a}=0 \tag{5.12}$$

from which we estimate the eigenvalues of (5.5). By ignoring the two equations that arise from the boundary conditions and using prior integrations twice we can write (5.11)

$$\hat{\mathbf{a}}+\lambda\hat{B}(I+mR)\mathbf{a}=0$$

where

$$\hat{\mathbf{a}}=[a_2\ a_3\ \dots\ a_N]^T.$$

Define

$$L_m:=\frac{1}{x^m}\frac{d}{dx}x^m\frac{d}{dx}$$

and the *Chebyshev coefficients of the operator* L_m as

$$L_m^{-1}\{T_i(2x-1)\}=\sum_{s=0}^{\infty}B_{i,s}^{(m)}\{T_i(2x-1)-1_{i,1}x-1_{i,0}\},\quad x\in[0,1]$$

where $1_{i,1}$, $1_{i,0}$ are chosen so that the boundary conditions in (5.5) are satisfied. The matrix

$$\hat{B}(I+mR)$$

defines the leading Chebyshev coefficients of the operator L_m. In this case the perturbed equation is given by

$$\frac{1}{x^m}\frac{d}{dx}x^m\frac{d}{dx}\{v_N+P\}+\lambda^*v_N=0$$

where

$$P(x)=\sum_{i=N+1}^{N+2}\sum_{s=N-1}^{N}B_{s,i}^{(m)}a_s[T_i(2x-1)-1_{i,1}x-1_{i,0}].$$

To compare the two approaches numerically we have considered the polar eigenvalue problem with $m=2$, $\gamma=1$ and $\delta=0$. The relative error in the first three eigenvalues is given in Table 2 for $N=8$ and $N=16$. We see from the table that the generalized Chebyshev method and the extended Dew & Scraton method are comparable when the perturbed form of the derivative is used in the boundary conditions. There is no advantage in applying the modification (5.1) to the generalized Chebyshev method. For completeness we have included the numerical results obtained using the Dew & Scraton method directly (i.e. extending the interval of integration to $[-1,1]$ and taking v as an even function). Although there is an improvement in the numerical results obtained the method cannot be easily generalized to a method based on piecewise Chebyshev expansions. We can conclude from these experiments that the approach used to derive the generalized Chebyshev method is competitive with existing Chebyshev methods. We have also confirmed this conclusion by comparing the numerical performance of these methods on several test parabolic equations.

TABLE 2

$$\frac{1}{x^2}\frac{d}{dx}\left\{x^2\frac{dv}{dx}\right\}+2v=0,\quad x\in[0,1],\qquad \frac{dv}{dx}=0\quad \textit{at } x=0 \textit{ and } v=0 \textit{ at } x=1$$

N	Errors	Exact boundary conditions (a) (i)	(iii)	(iv)	Approx. boundary conditions (b) (i)	(ii)	(iii)
8	$E(\lambda_1)$	$0{\cdot}331E{-}7$	$0{\cdot}734E{-}5$	$0{\cdot}943E{-}13$	$0{\cdot}353E{-}7$	$0{\cdot}122E{-}6$	$0{\cdot}230E{-}7$
	$E(\lambda_2)$	$0{\cdot}448E{-}5$	$0{\cdot}696E{-}5$	$0{\cdot}172E{-}7$	$0{\cdot}432E{-}5$	$0{\cdot}437E{-}5$	$0{\cdot}699E{-}5$
	$E(\lambda_3)$	$0{\cdot}301E{-}3$	$0{\cdot}555E{-}2$	$0{\cdot}129E{-}4$	$0{\cdot}359E{-}3$	$0{\cdot}145E{-}2$	$0{\cdot}120E{-}2$
16	$E(\lambda_1)$	$0{\cdot}652E{-}13$	$0{\cdot}312E{-}10$	$0{\cdot}569E{-}14$	$0{\cdot}401E{-}13$	$0{\cdot}620E{-}13$	$0{\cdot}724E{-}14$
	$E(\lambda_2)$	$0{\cdot}358E{-}13$	$0{\cdot}288E{-}14$	$0{\cdot}198E{-}14$	$0{\cdot}809E{-}14$	$0{\cdot}142E{-}12$	$0{\cdot}927E{-}13$
	$E(\lambda_3)$	$0{\cdot}108E{-}9$	$0{\cdot}218E{-}7$	$0{\cdot}113E{-}13$	$0{\cdot}114E{-}9$	$0{\cdot}923E{-}9$	$0{\cdot}858E{-}9$

(i) Generalized Chebyshev method, (ii) generalized Chebyshev method with modification (5.1), (iii) extended Dew & Scraton method and (iv) Dew & Scraton method on $[-1, 1]$.

6. Numerical Example

To illustrate the numerical performance of the global element method we have considered the following problem taken from Bakker (1977);

$$\begin{aligned}\frac{\partial u}{\partial t}&=5x^{-2}\frac{\partial}{\partial x}\left\{x^2\frac{\partial u}{\partial x}\right\}-1000e^u, && x\in[0,\tfrac{1}{2})\\ &\;x^{-2}\frac{\partial}{\partial x}\left\{x^2\frac{\partial u}{\partial x}\right\}-e^u, && x\in(\tfrac{1}{2},1]\end{aligned}\tag{6.1}$$

subject to the boundary conditions

$$\frac{\partial u}{\partial x}=0\quad\text{at } x=0,\qquad u(1,t)=1,\quad\text{for } t>0$$

and initial condition

$$u(x,0)=0.$$

We assume the internal boundary condition

$$5\lim_{x\to\frac{1}{2}^-}\frac{\partial u}{\partial x}=\lim_{x\to\frac{1}{2}^+}\frac{\partial u}{\partial x}.$$

There is a discontinuity between the boundary condition and the initial condition at $x=1$.

We have compared the numerical solution obtained by the

(i) finite difference code given in Sincovec & Madsen (1975),
(ii) finite element code written by Bakker using linear basis functions.

In each case the time integration is performed using Gear's method with a local error tolerance equal to 10^{-7} using a mixed error test.

To measure the error we first compute the solution using a high precision run. The error can then be measured using the norm

$$E(t) = \sqrt{\int_0^1 x^2[U(x,t)-u(x,t)]^2\,dx} \quad \text{for } t \in (0, 0{\cdot}3]$$

where $U(x, t)$ is the computed solution at the point (x, t) and the integral was evaluated using the trapezium rule with 100 points.

Figure 1 shows how $E(t)$ varies with t. We have chosen 41 and 81 equally spaced mesh points for the finite difference and finite element codes. For the generalized Chebyshev method we have used two elements $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ with $N_1 = N_2 = 7$ (15 mesh points) and four equally spaced elements with $N_j = 9$ on each element (37 mesh points). We can see from the graphs given in Fig. 1 that very satisfactory results are obtained using the Chebyshev method. As we expect the discontinuity in the boundary and initial condition causes a large error initially which rapidly dies away.

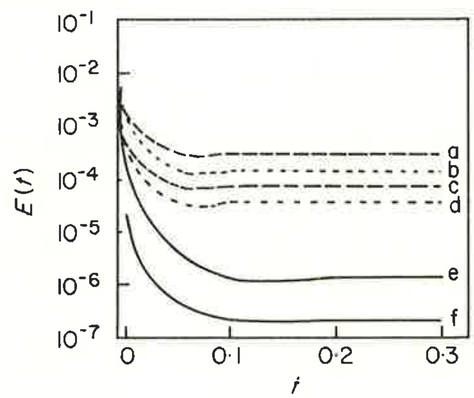

FIG. 1. Graph of $E(t)$ for the test problem (6.1). (a) Finite differences ($N = 41$); (b) finite elements ($N = 41$); (c) finite differences ($N = 81$); (d) finite elements ($N = 81$); (e) Chebyshev method ($N = 15$, 2 elements); (f) Chebyshev method ($N = 37$, 4 elements), N denotes the number of mesh points (i.e. number of ODEs to solve).

7. Conclusion

We have shown in this paper that the Chebyshev methods can be extended to handle general parabolic equations and that the generalized Chebyshev method compares very favourably with the finite element (linear basis function) method and the finite difference method. An advantage of the Chebyshev approach is that it is easy to vary the size of an element and the degree of the polynomial used on each element. It remains an interesting problem to see if the perturbation term can be used to select the size of each element and the degree of the polynomial automatically in some optimum manner.

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