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WITH CORRECTIONS.

A Note on the Extension of the Chebyshev Method to Quasi-Linear Parabolic P.D.E.s with Mixed Boundary Conditions

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Dew [1] proposed a method for computing the numerical solution to quasi-linear parabolic p.d.e.s using a Chebyshev method. The purpose of this note is to extend the method to problems with mixed boundary conditions. An error analysis for the linear problem is given and a global element Chebyshev method is described.

KEY WORDS: Parabolic P.D.E., quasi-linear, Chebyshev method.

C.R. CATEGORIES: 5.17.

1. INTRODUCTION

In a recent paper, Dew [1], one of the authors described an algorithm based on Chebyshev polynomials to compute the numerical solution to parabolic p.d.e.s of the type

$$\frac{\partial^2 u}{\partial x^2}=\sigma(u,x,t)\frac{\partial u}{\partial t}+f(u,x,t),\ (x,t)\in[-1,1]x(0,T] \tag{1.1}$$

subject to

$$u(-1,t)=g_{-1}(t),\ u(1,t)=g_1(t),\qquad t\in(0,T] \tag{1.2}$$

and

$$u(x,0)=K(x),\qquad x\in[-1,1]. \tag{1.3}$$

The purpose of this paper is to extend the method and error analysis to problems with mixed boundary conditions of the type

$$\alpha u(-1,t)+\beta\frac{\partial}{\partial x}u(-1,t)=g_{-1}(t)$$

and

$$\gamma u(1,t)+\delta\frac{\partial}{\partial x}u(1,t)=g_1(t). \tag{1.4}$$

Following the standard Chebyshev method, the $u(x,t)$ is approximated by a polynomial of degree N written in the form

$$U_N(x,t)=\sum_{i=0}^{N} a_i(t)T_i(x) \tag{1.5}$$

where $T_i(x)$ is the Chebyshev polynomial of degree i, and the function

$$Q(x,t):=\sigma(U_N,x,t)\frac{\partial U_N}{\partial t}+f(U_N,x,t) \tag{1.6}$$

is approximated by a polynomial of degree N that interpolates $Q(x,t)$ at the Chebyshev points

$$X_N:=\left\{x_i=\cos\left(\frac{i\pi}{N}\right), \qquad i=0,1,\ldots,N\right\}.$$

We write this polynomial as

$$Q_N(x,t)=\sum_{i=0}^{N} q_i(t)T_i(x) \tag{1.7}$$

The coefficients $a_i(t)$ are then chosen so that the following equations are satisfied

$$\sum_{i=0}^{N}(-1)^i(\alpha-i^2\beta)a_i(t)=g_{-1}(t)$$

$$\sum_{i=0}^{N}(\gamma+\delta i^2)a_i(t)=g_1(t) \tag{1.8}$$

and

$$a_i(t)=\sum_{s=0}^{N}A_{i,s}q_s(t), \qquad i=2,3,\ldots,N$$

where the coefficients $A_{i,s}$ are defined in Dew [1].

For sufficiently simple problems it is possible to eliminate $q_0(t)$ and $q_1(t)$. When

$$D = \gamma(\alpha - \beta) + \alpha(\gamma + \delta) \neq 0 \tag{1.9}$$

it is then possible to use an obvious extension of the method described in Dew [1].

Alternatively we can consider (1.8) as a system of algebraic and ordinary differential equations and compute the numerical solution directly using an o.d.e. solver adapted to solve equations of the form $\phi(t, y, y') = 0$ (in standard notation). Gear's method, for example, can readily be implemented in this form. Equation (1.8) can conveniently be mapped into the solution values at the Chebyshev points by using the mapping matrix Ω whose (i,j)th element is defined as

$$\Omega_{i,j} := T_i(x_j), \qquad x_j \in X_N. \tag{1.10}$$

The inverse of Ω can be determined analytically and is therefore more satisfactory than the mapping matrix proposed in Dew [1]. Implementing (1.8) in this form has the advantage that the problem specification can be generalized. (For example, the method can be extended to boundary conditions of the form

$$\hat{g}_i\left(u(-1,t), u(1,t), \frac{\partial u(-1,t)}{\partial x}, \frac{\partial u(1,t)}{\partial x}, t\right) = 0 \quad i = 1, 2.)$$

Numerical experiments indicate however that it is more satisfactory to solve, where possible, an explicit system of ordinary differential equations. In the next section we shall consider a new algorithm which reduces the differential equation to an explicit system of ordinary differential equations when there are derivative boundary conditions. An error analysis for the linear problem shows that the new algorithm is likely to lead, for sufficiently large N, to a more accurate solution than the solution obtained using (1.8).

The algorithm is then extended to a global element method which can be used when the solution $u(x,t)$ cannot be adequately represented by a polynomial defined on $[-1, 1]$ for each $t \in (0, T]$.

2. AN IMPROVED ALGORITHM

Define

$$R(u, x, t) := \sigma(u, x, t)\frac{\partial u}{\partial t} + f(u, x, t) \tag{2.1}$$

then an improved algorithm can be derived by noting that the derivative $\partial u/\partial x$ can be estimated from the formula

$$\frac{\partial u}{\partial x}=\int^{x} R(u,x,t)dx+A \tag{2.2}$$

where

$$A=\tfrac{1}{2}(u(1,t)-u(-1,t)-(H(1,t)-H(-1,t))) \tag{2.3}$$

and

$$H(x,t)=\int^{x}\int^{x} R(u,x,t)dx\,dx. \tag{2.4}$$

The integrals appearing in the above expression are indefinite integrals. The solution $u(x,t)$ is again approximated by a polynomial of degree N written as

$$U_N^T(x,t)=\sum_{i=0}^{N} a_i^T(t)T_i(x) \tag{2.5}$$

and

$$\bar{Q}(x,t):=R(U_N^T,x,t). \tag{2.6}$$

The coefficients $\{a_i^T(t)\}$ are chosen so that they satisfy the equations

$$\left.\begin{aligned} a_i^T(t)&=\sum_{s=0}^{N} A_{i,s}\bar{q}_s, \qquad i=2,3,\ldots,N\\ \sum_{i=0}^{N}((-1)^i\alpha+\tfrac{1}{2}\beta(1-(-1)^i)a_i^T(t)&+\beta(e_i^{(-1)}-E_i)\bar{q}_i(t)=Z^{(-1)}\\ \sum_{i=0}^{N}(\gamma+\tfrac{1}{2}\delta(1-(-1)^i)a_i^T(t)&+\delta(e_i^{(1)}-E_i)\bar{q}_i(t)=Z^{(1)} \end{aligned}\right\} \tag{2.7}$$

where the coefficients $\{\bar{q}_i\}$ are the Chebyshev coefficients for $\bar{Q}(x,t)$ interpolated at the Chebyshev points, $x_i \in X_N$, and

$$e_i^{(y)}=\left[\int^{x} T_i(x)dx\right]_{x=y}, \quad E_i=\tfrac{1}{2}(1-(-1)^i)\left[\int^{x}\int^{x} T_i(x)dx\right]_{x=1}.$$

The parameter Z is defined as

$$Z^{(y)}=\sum_{s=N-1}^{N} A_{s+2,s}\bar{q}_s(t)z_s^{(y)}+g_y(t), \quad y=\pm 1 \tag{2.8}$$

where $z_s^{(y)}$ is given in Section 3.

The coefficients $\{a_i^T, \bar{q}_i\}$ can then be mapped into the solution values evaluated at the Chebyshev points using the mapping matrix Ω defined by Eq. (1.11). Equations (2.7) define a system of first order o.d.e.s when β and/or $\delta \neq 0$, which can be written (if desired) in normal form.

A feature of Chebyshev methods is that it is easy to obtain the perturbed form of the differential equation that $U_N(x,t)$ exactly satisfies. A similar result can be shown for the improved algorithm. Define the perturbation function as

$$P_N(x,t) := \sum_{s=N+1}^{N+2} \bar{q}_{s-2} A_{s,s-2} T_s(x). \tag{2.10}$$

We can then prove the following lemma:

LEMMA 1 *The approximate solution $U_N^T(x,t)$ whose coefficients $\{a_i^T\}$ are defined by Eq. (2.7) is an exact solution of the perturbed equation*

$$\frac{\partial^2}{\partial x^2}\{U_N^T(x,t)\} = \bar{Q}_N(x,t) - \frac{\partial^2}{\partial x^2}\{P_N(x,t)\},\ (x,t) \in [-1,1] x (0,T]$$

subject to

$$\alpha U_N^T(-1,t) + \beta \frac{\partial}{\partial x} U_N^T(-1,t) = -\beta\phi(-1,t) + Z^{(-1)}(t) + g_{-1}(t)\, t \in (0,T]$$

$$\gamma U_N^T(1,t) + \delta \frac{\partial}{\partial x} U_N^T(1,t) = -\delta\phi(1,t) + Z^{(1)}(t) + g_1(t)$$

and

$$U_N^T(x,0) = K_N(x), \quad x \in [-1,1]$$

where

$$\phi_N(x,t) = \frac{\partial}{\partial x}\{P_N(x,t)\} - \frac{P_N(1,t) - P_N(-1,t)}{2}$$

and $\bar{Q}_N(x,t)$, $K_N(x)$ interpolate the functions $\bar{Q}(x,t)$, $K(x)$ respectively at the Chebyshev points, $x_i \in X_N$.

Proof The proof follows on substitution of the series expansion for $U_N^T(x,t)$, $Q_N(x,t)$ and then on integrating in the standard manner.

This Lemma is the starting point for the error analysis. By choosing

$$Z^{(-1)}(t) = \beta\phi(-1,t) \text{ and } Z^{(1)}(t) = \delta\phi(1,t) \tag{2.11}$$

it is easily seen that $U_N^T(x,t) \equiv U_N(x,t)$.

It is helpful to introduce operator notation. Following Dew [1] we define F as the space of function defined on $[-1,1]\times[0,T]$ such that

$$f_g(.,t)\in L_2[-1,1] \text{ for each fixed } t\in[0,T]$$

and consider the norm

$$f_g(.,t)=\sqrt{\int_{-1}^{1}[f_g(x,t)]^2\,dx}.$$

Define $H\subset F$ such that all functions belonging to H satisfy the homogeneous boundary conditions (1.4) and have at least piecewise continuous second derivatives in the first variable x. The differential equation can then be written in the operator notation: $L: H\to F$

$$Lu_g=R \text{ for each } t\in(0,T] \tag{2.12}$$

where $u(x,t)=u_g(x,t)+G(x,t)$, and $G(x,t)$ is chosen so that $u(x,t)$ satisfies the boundary conditions (1.4). For $D\neq 0$ the inverse operator $L^{-1}:F\to H$ exists for each $t\in(0,T]$.

3. AN ERROR ANALYSIS FOR THE LINEAR BOUNDARY VALUE PROBLEM

The nature of the approximation can most clearly be seen by considering the differential equation

$$\frac{\partial^2 u}{\partial x^2}=f(x,t), \quad x\in[-1,1] \text{ for each } t\in(0,T] \tag{3.1}$$

subject to the boundary conditions (1.4). In this case

$$Q(x,t)=\bar{Q}(x,t)=f(x,t)=\sum_{i=0}^{\infty} f_i(t)T_i(x)$$

and

$$Q_N(x,t)=\bar{Q}_N(x,t)=f_N(x,t)=\sum_{i=0}^{N} f_i^{(N)}(t)T_i(x).$$

The following theorem enables us to compare the accuracy of the two methods defined by Eq. (1.8) and (2.7) respectively. Define

$$P_q(x,t)=\sum_{s=N+1}^{N+2} A_{s,s-2}q_{s-2}\{T_s(x)-1_{1,s}x-1_{0,s}\} \tag{3.2}$$

where the coefficients $1_{1,s}$ and $1_{0,s}$ are given by

$$\begin{bmatrix} \alpha & \beta-\alpha \\ \gamma & \delta+\gamma \end{bmatrix}\begin{bmatrix} 1_{0,s} \\ 1_{1,s} \end{bmatrix}=\begin{bmatrix} (\alpha-\beta s^2)(-1)^s \\ \gamma+\delta s^2 \end{bmatrix}$$

and

$$\bar{P}_q(x,t)=\sum_{s=N+1}^{N+2} A_{s,s-2}q_{s-2}\{T_s(x)-L_{1,s}x-L_{0,s}\} \tag{3.3}$$

where the coefficients $L_{1,s}$ and $L_{0,s}$ are given by

$$\begin{bmatrix} \alpha & (\beta-\alpha) \\ \gamma & (\delta+\gamma) \end{bmatrix}\begin{bmatrix} L_{0,s} \\ L_{1,s} \end{bmatrix}=\begin{bmatrix} +\alpha(-1)^s+\beta\dfrac{(1-(-1)^s)}{2}+z_s^{(-1)} \\ +\gamma \qquad +\delta\dfrac{(1-(-1)^s)}{2}+z_s^{(1)} \end{bmatrix}.$$

THEOREM 1 *Suppose that the function $u(x,t)$ satisfies* (3.1) *subject to the boundary conditions* (1.4) *for each $t(0,T]$, $f\in F$ and that $D\neq 0$. Then the following statements are true:*

i) If $U_N(x,t)$ is the function whose coefficients $\{a_i(t)\}$ satisfy (1.8) *with $q_i=f_i^{(N)}$ then*

$$u(x,t)-U_N(x,t)=P_{f^{(N)}}(x,t)+L^{-1}\{f-f_N\} \tag{3.4}$$

ii) If $U_N^T(x,t)$ is the function whose coefficients $\{a_i^T\}$ satisfy (2.7) *with $q_i=f_i^{(N)}$ then*

$$u(x,t)-U_N^T(x,t)=\bar{P}_{f^{(N)}}(x,t)+L^{-1}\{f-f_N\}. \tag{3.5}$$

Proof The result for (i) follows directly from the perturbed equation for $U_N(x,t)$. For (ii) write

$$\hat{U}_N^T=U_N^T+L_1x+L_0,\quad \hat{U}_N^T\in H,$$

and use the result of Lemma 1.

From the result of the Theorem the parameter $z_s^{(\pm i)}$ can be chosen as

$$z_s^{(-1)}=\alpha(-1)^s-\beta\frac{(1-(-1)^s)}{2}$$

and

$$z_s^{(1)}=\gamma-\delta\frac{(1-(-1)^s)}{2}$$

providing that β and/or δ are nonzero. If $\beta=\delta=0$ then it is necessary to ensure that the boundary conditions are exactly satisfied and hence

$$z_s^{(1)} = z_s^{(-1)} = 0.$$

The main advantage of introducing $z_s^{(\pm 1)}$ is that it simplifies the expression for the perturbation term which makes it easier to compute error estimates.

The advantage of the improved algorithm can be seen from the result of the theorem, since the coefficients $\{1_{i,s}\}$ are of $O(s^2)$ whereas the coefficients $\{L_{i,s}\}$ are of $O(1)$. Of course for the simple linear problem we can further improve the solution by adding the perturbation term to the computed solution with an error

$$L^{-1}\{f - f_N\}$$

for both methods. We now show that the perturbation term is the dominant part of the error when the coefficients $\{f_i\}$ converge rapidly.

LEMMA 2 *Suppose that $D \neq 0$ and $w(t)$ is a continuous function then*

$$\|L^{-1}w(t)T_N(x)\|_x = |w(t)|O(1/N^2), \quad N > 2. \tag{3.7}$$

Proof Is is easily shown that

$$L^{-1}w(t)T_N(x) = w(t) \sum_{s=N-2}^{N+2} A_{s,N}\{T_s(x) - 1_{1,s}x - 1_{0,s}\}. \tag{3.8}$$

Let

$$L_{1,N}^{(A)} = \sum_{s=N-2}^{N+2} A_{s,N} 1_{1,s}, \quad L_{0,N}^{(A)} = \sum_{s=N-2}^{N+2} A_{s,N} 1_{0,s}$$

then

$$\begin{bmatrix} \alpha+\beta & -\alpha \\ \gamma+\delta & +\gamma \end{bmatrix} \begin{bmatrix} L_{0,N}^{(A)} \\ L_{1,N}^{(A)} \end{bmatrix} = \begin{bmatrix} \sum_{s=N-2}^{N+2} A_{s,N}(\alpha - \beta s^2)(-1)^s \\ \sum_{s=N-2}^{N+2} A_{s,N}(\gamma + \delta s^2) \end{bmatrix}.$$

But

$$\sum_{s=N-2}^{N+2} A_{s,N}s^2 = \frac{-1}{(N^2-1)}, \quad \sum_{s=N-2}^{N+2} A_{N,s} = \frac{3}{(N^2-1)(N^2-4)}$$

and since $D \neq 0$ it follows immediately that for $N > 2$

$$L_{0,N}^{(A)} = O(1/N^2), \quad L_{1,N}^{(A)} = O(1/N^2)$$

The result (3.7) then follows directly from (3.8).

If we assume that the Chebyshev coefficients $\{f_i\}$ for the function $f(x,t)$ converge sufficiently rapidly that $f_{N+2}, f_{N+3}, \ldots$ can be ignored then the error

$$E_f(x,t) = f(x,t) - f_N(x,t) \approx f_{N+1}\{T_{N+1}(x) - T_{N-1}(x)\}$$

(see Fox and Parker (1968)) and therefore from the result of Lemma 2 the perturbation term will dominate and hence E_p is a good estimate of the error $\|u(.,t) - U_N(.,t)\|_x$. If we assume that the exact Chebyshev coefficients of $f(x,t)$ are used (i.e. $q_i = \bar{q}_i = f_i$) then

$$\|u(.,t) - U_N(.,t)\|_x \leqq C_1 \{\max |f_{N-1}|, |f_{N+1}|\} + N^{-2} \sum_{i=N+1}^{\infty} |f_i|$$

$$\|u(.,t) - U_N^T(.,t)\|_x \leqq C_2 N^{-2} \sum_{i=N-1}^{\infty} |f_i|, \qquad N > 2.$$

where C_1 and C_2 are constants.

4. AN ERROR ANALYSIS FOR THE LINEAR PARABOLIC P.D.E.

The error analysis can be extended to the linear parabolic p.d.e. of the form

$$\frac{\partial^2 u}{\partial x^2} = \sigma \frac{\partial u}{\partial t} + f(x,t), \quad \sigma > 0, \quad (x,t) \in [-1,1]x(0,T] \tag{4.1}$$

where σ is a constant, subject to the mixed boundary conditions (1.4). In the following analysis we shall assume that

$$\alpha \neq 0, \; \gamma \neq 0, \; \beta/\alpha < 0, \; \delta/\gamma > 0 \tag{4.2}$$

which ensures that $D \neq 0$ and the operator L (see (2.11)) is negative definite. In addition we shall assume that the error, $e(x,t)$, in the numerical solution (using either (1.8) or (2.7)) satisfies the inequality

$$\int_{-1}^{1} e \frac{\partial e}{\partial t} dt < 0 \qquad \text{for all } t \in (0,T] \tag{4.3}$$

so that

$$\left\|\frac{\partial e}{\partial t}\right\|_x = -\bar{\gamma}(t) \frac{d}{dt} \|e(.,t)\|_x, \; \|e(.,t)\|_x \neq 0 \tag{4.3}$$

where $\bar{\gamma}(t) \geqq 1$ is assumed to be a bounded function ($\bar{\gamma}(t) < \hat{\gamma}$). Under these assumptions the analysis of Dew [1] follows directly and

$$\|e(.,t\|_x \leqq \exp\left(\frac{-\lambda_1 t}{\sigma\hat{\gamma}}\right)\|K(.-K_N(.)\|_x + \max_{t\in(0,T]}\{\|P(.,t)\|_x + \|L^{-1}(f-f_N)\|_x\} \tag{4.4}$$

where

$$P(x,t) = P_q(x,t) \text{ for the method given by Eqs. (1.8)}$$
$$= \bar{P}_{\bar{q}}(x,t) \text{ for the method given by Eqs. (2.7)}$$

and λ_1 is the fundamental eigenvalue of the restricted operator $\bar{L}^{-1}$: $H \to H$.

From the results of the previous section, we have that

$$\|P_q(.,t)\|_x = \max(\sigma|a'_{N-1}| + |f_{N-1}|, \sigma|a'_N| + |f_N|)\cdot O(1)$$

and

$$\|\bar{P}_q(.,t)\|_x = \max(\sigma|a^{T'}_{N-1}| + |{}^f{}_{N-1}|, \sigma|a^{T'}_N| + |f_N|)\cdot O(N^{-2})$$

where the dash denotes differentiation w.r.t. t. This algorithm is likely to be more satisfactory providing that the initial error $\|K(.)-K_N(.)\|_x$ does not dominate the solution. It should be noted that Dew and Scraton [2] obtained a similar improvement for the heat equation as shown in Scraton [4]. However, it is more satisfactory to compute a numerical solution using Eqs. (2.7) than the Dew and Scraton method and also the new method can be used even if D (1.9) is zero.

5. GLOBAL ELEMENT CHEBYSHEV METHOD

For a number of parabolic p.d.e.s arising in practice it is not sufficient to approximate the solution by a polynomial defined on the interval $[-1, 1]$ for each t. In such cases a global element Chebyshev method can be used. That is we partition the interval $[-1, 1]$ into elements $\{I_m := [y_m, y_{m+1}], m = 1, 2, \ldots, M, y_1 = -1, y_M = 1, y_m < y_{m+1}\}$ and write the solution as

$$u(x,t) = \bigcup_{m=1}^{M} u^m(x,t) \tag{5.1}$$

where

$$u^m(x,t) := u(x,t), \ x \in I_m \text{ and zero elsewhere.}$$

Define $W_m: I_m \to [-1,1]$ and approximate the solution by

$$U_N^G(x,t) = \bigcup_{m=1}^{M} U_N^m(x,t) \tag{5.2}$$

where

$$U_N^m(x,t) = \sum_{i=0}^{N_m} a_{m,i} T_i(W_m(x)), \quad x \in I_m,$$
$$= 0 \quad \text{elsewhere.}$$

At the internal nodes we impose the boundary conditions

$$\left.\begin{aligned} U_N^m(x_m,t) &= U_N^{m+1}(x_m,t) \\ \text{and} \quad \frac{\partial U_N^m}{\partial x}(x_m,t) &= \frac{\partial U_N^{m+1}}{\partial x}(x_m,t) \end{aligned}\right\} \quad m = 2,3,\ldots,M.$$

We assume that the function R (defined in Eq. (2.1)) is a continuous function in its arguments u and t for all $t \in (0,T]$ and $u \in F$ and that it is a piecewise continuous function for the x variable, $x \in [-1,1]$. Any discontinuities in R must be at the internal nodes. We further assume that

$$Q^G(x,t) = R(U_N^G, x, t)$$

can be adequately represented by a piecewise polynomial

$$Q_N^G(x,t) = \bigcup_{m=1}^{M} Q_N^m(x,t)$$

where

$$Q_N^m(x,t) = \sum_{i=0}^{N_m} q_{m,i} T_i(W_m(x)), \quad x \in I_m$$
$$= 0 \text{ elsewhere.}$$

where $Q_N^m(x,t)$ interpolates $Q_N^G(x,t)$ at the points $\bar{X}_m := \{W_m(x_i), x_i \in X_{N_m}\}$. The algorithm given in Section 2 can then be extended. That is the coefficients $\{a_{m,i}\}$ are chosen to satisfy the equations

$$a_{m,i} = \mu_m^2 \sum_{s=0}^{N_m} A_{i,s} q_{m,s}, \quad \begin{aligned} m &= 1,2,\ldots,M \\ i &= 2,\ldots,N_m \end{aligned}$$

$$\sum_{i=0}^{N_1} (\alpha + \tfrac{1}{2}\beta(1-(-1)^i)a_{1,i} + \mu_1 \beta(e_i^{(-1)} - E_i)q_{1,i}) = g_{-1}(t)$$

$$\sum_{i=0}^{N_M} (\gamma + \tfrac{1}{2}\delta(1-(-i)^i)a_{M,i} + \mu_M \delta(e_i^{(1)} - E_i)q_{M,i}) = g_1(t)$$

together with the condition for continuity of the derivative at the internal node

$$\sum_{i=0}^{N_m} \tfrac{1}{2}(1-(-1)^i)a_{m,i}+\mu_m(e_i^{(-1)}-E_i)q_{m,i}$$
$$-\sum_{i=0}^{N_{m+1}} \tfrac{1}{2}(1-(-1)^i)a_{m+1,i}+\mu_{m+1}(e_i^{(1)}-E_i)q_{m+1}=0$$
$$m=2,3,\ldots,M.$$

where

$$\mu_m=(y_{m+1}-y_m)/2$$

The coefficients $\{a_{m,i}, q_{m,i}\}$ can be mapped into the solution values at the points X_m using the mapping matrix Ω (1.11) defined on each element such that the continuity of $U_N^G(x,t)$ and $Q^G(x,t)$ is preserved at the internal nodes. The algorithm requires a slight modification in the case when R is only piecewise continuous at the internal nodes. If β and/or $\delta \neq 0$ then (5.3), (5.4) mapped into the solution values defines a system of ordinary differential equation which can be solved by Gear's method. The Jacobian matrix arising in Gear's method is banded with a maximum bandwidth of $\max_m(N_m+N_{m+1}+1)$. In the case when $\beta=\delta=0$ (5.3), (5.4) can be reduced to a system of ordinary differential equations and two algebraic equations.

It is easily shown that the function $U_N^G(x,t)$ satisfies a perturbed form of the original differential equation. The error analysis given in this paper and the techniques for estimating the error described in Dew [1] can be extended to the global element method. In particular it is possible to estimate the error in the solution across each element.

6. NUMERICAL EXAMPLE

To compare the relative accuracy of the solution $U_N(x,t)$ and $U_N^T(x,t)$ whose coefficients $\{a_i\}$ satisfy Eqs. (1.8) and (2.7) respectively we have considered the parabolic p.d.e.

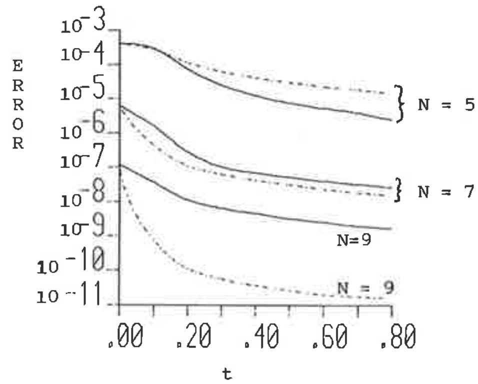
$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+x^2e^{-u}, \qquad (x,t)\in[0,1]x(0,.8]$$

subject to

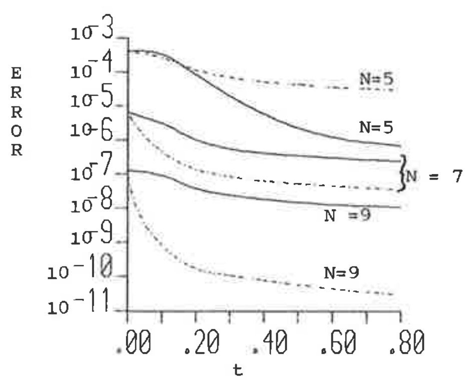
$$\frac{\partial u(0,t)}{\partial x}=0, \quad t\in(0,.8], \quad u(x,0)=\log_e(\tfrac{1}{2}x^2+0.1)$$

TABLE 1

The error curves for problems A and B



PROBLEM A - mixed boundary conditions



PROBLEM B - Neumann boundary conditions

The solid line ——— denotes the error $\|U_N(.,t)-u(.,t)\|_x$

The dotted line ----- denotes the error $\|U_N^T(.,t)-u(.,t)\|_x$.

and either

PROBLEM A—mixed boundary conditions

$$u(1,t)+(0.6+t)\frac{\partial u(1,t)}{\partial x}=\log_e(t+0.6)+1, t\in(0,.8]$$

or

PROBLEM B—Neumann boundary conditions

$$\frac{\partial u(1,t)}{\partial x}=\frac{1}{(0.6+t)} \qquad t\in(0,.8]$$

The exact solution is $u(x,t)$

$$u(x,t)=\log_e(\tfrac{1}{2}x^2+t+0.1).$$

It was not practical to compute a numerical solution directly from Eq. (1.8) because the iterative procedure in Gear's method failed to converge. Hence to compare the relative accuracy of the two solutions we used Eq. (2.7)

$Z^{(\pm 1)}$ as given in Eq. (2.11) (coefficients define $U_N(x,t)$)

and

$Z^{(\pm 1)}=0$ (coefficients define $U_N^T(x,t)$)

Graphs of the error norms $\|U_N(.,t)-u(.,t)\|_x$ and $\|U_N^T(.,,t)-u(.,t)\|_x$ against t for $N=5$, 7 and 9 and for problems A and B are given in Table 1. The numerical solution of Eq. (2.7) was computed using Gear's methods with a local error tolerance $0.5_{10}-8$. The improvement in accuracy obtained using the new algorithm can be seen for $N>5$.

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