Inhibition of the Measurement of the Wavefunction of a Single Quantum System and the Projection Postulate

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INTRODUCTION

Recently, with the technological developments which allow manipulation of single quantum systems, such as atoms, electrons and photons, in laboratories all over the world, there has been an interest in the quantum theory of a single system. This interest raises several fundamental questions, one of which is the question of the meaning of the quantum wavefunction. Quantum mechanics describes a single system by a corresponding wavefunction. The wavefunction contains all relevant information about the single physical system. However, in order to obtain this information, and determine the wavefunction, one needs to consider the statistics of the results of measurements performed on an ensemble of identical systems. Therefore, the wavefunction is said to have a statistical (or epistemological) meaning. Naturally, one would ask the question: Can we give the quantum wavefunction a deeper physical (or ontological) meaning? Aharonov, Anandan and Vaidman (1,2) showed recently that the wavefunction of a single system could be determined from the results of a series of "protective measurements" performed on the system. In the protective measurement scheme, a-priori knowledge of the wavefunction of the system is used in order to measure this system and protect its wavefunction from changing at the same time. Aharonov, Anandan and Vaidman argued that the protective measurement accounts for the physical reality of the wavefunction. Yet, it seems that one should be able to measure the wavefunction of a single system without any a-priori knowledge, if the wavefunction were real. Considering a measurement of the spin of a single spin-1/2 system, with an unknown wavefunction, Royer showed (3) recently that this measurement can be physically reversed, so that the system is back in its unknown initial state, with a finite success probability. However, as Huttner showed (4), the statistics of a series of successful "reversible measurements", performed on a single spin-1/2 system, are independent of the initial spin wavefunction of the system. Therefore, reversible measurements cannot be used for the measurement of the wavefunction of a single system.

According to the projection postulate, a precise measurement performed on a single system would always yield one of the eigenvalues of the measured observable. Usually, this eigenvalue can be used to estimate the expectation value of the observable, i.e., the center position of the probability density of the observable. But the uncertainty associated with the measured observable, i.e., the width of the probability density, could never be estimated using this single measurement result. In fact, this uncertainty cannot be estimated even if we use the results of additional measurements performed on the single system. After the measurement, the wavefunction of the system collapses to the eigenstate, which corresponds to the measured eigenvalue. Therefore, the results of additional measurements performed on the system would not add any information about the initial wavefunction of the system. Now, consider a very weak measurement performed on the single system. This measurement would leave the wavefunction of the system almost unchanged. Consecutive weak measurements performed on the same system, therefore, would give us some additional information about the initial wavefunction of the system. Since the measured system is approximately in the same state when the different measurements are performed, one may expect the statistics of the measurement results to be approximately the same as the results of measurements performed on an ensemble. Specifically, one may expect the statistics of these measurement results to enable us to estimate the uncertainties associated with the measured observables with finite estimate errors. It is the ability to estimate the uncertainty associated with a specific observable, i.e., the width of the probability density of this observable, which distinguishes a measurement of the wavefunction from a measurement of the observable. In this work, we show that this intuitive picture fails, and one cannot, in fact, extract any information about the initial wavefunction of a single system at all, using repeated weak quantum measurements.

First, considering the general model of repeated quantum measurements performed on a single system, we prove (5) that the statistics of the results of these measurements are independent of the initial uncertainties associated with the measured observables. We show that this is a direct result of the projection postulate. Therefore, the physical mechanism which is responsible for the inhibition of the measurement of the wavefunction of a single quantum system, is the also responsible for the change of the wavefunction due to the measurements. To illustrate this result, we discuss two specific examples, both using quantum non-demolition (6,7) measurements, which have been demonstrated experimentally. The first example (8) is that of repeated quantum non-demolition (QND) measurements of the photon-number (9-12), performed on a squeezed state of light (which includes a coherent state as a special case), i.e., a generalized minimum uncertainty state. We show that due to the measurement process, the wavefunction of this squeezed state undergoes saturated quantum Brownian motion, while it continuously collapses to a photon-number eigenstate. The second example is that of alternating QND measurements of the two (slowly varying) quadrature amplitudes (13,14) of a squeezed state of light. We show that the noise distribution of this squeezed state is continuously modified by the measurements, until it is determined solely by the measurement errors, and is completely independent of the initial intrinsic uncertainties of the two quadrature amplitudes. In this limit the squeezed state undergoes free diffusion, preserving its noise distribution. The changes in the measured squeezed state of light are different in each example. Yet, in both examples, it is due to these changes that the statistics of the measurement results lack any information about the uncertainties of the measured observables, and the measurement of the wavefunction of the single squeezed state is inhibited. Note that the QND measurement of the photon-number is analogous to the QND measurement of the momentum of a free particle (15). Also, the QND measurements of the two quadrature amplitudes are mathematically equivalent to the QND measurements of the position and the momentum of a mechanical harmonic oscillator. Our two examples, therefore, cover all QND measurements known today.

THE GENERAL PROOF: REPEATED QUANTUM MEASUREMENTS CANNOT INFER THE UNKNOWN WAVEFUNCTION OF A SINGLE SYSTEM

Consider a quantum measurement of the observable $\hat{q}$ of a single system. This system, the signal, is initially in the pure state $|\psi\rangle_s$, and is described by the density operator $\hat{\rho}_0 = |\psi\rangle_s\, {}_s\langle\psi|$. The only requirement of the measurement is that it fulfills the generalized projection postulate. Sometimes these measurements are referred to as "Pauli's first-kind measurements". During the measurement process, the signal is correlated to a probe system, and after the correlation the probe is measured to yield the inferred measurement result $\tilde{q}_1$ (Fig. 1). The probability-amplitude operator, $\hat{Y} = {}_p\langle\tilde{q}_1|\hat{U}|\phi\rangle_p$, completely describes the three stages of this measurement (16): The preparation of the probe in the pure state $|\phi\rangle_p$, the interaction of the probe with the signal, $\hat{U}$, and the result of the measurement, $\tilde{q}_1$, which corresponds to the state of the probe after the measurement, $|\tilde{q}_1\rangle_p$. The probability of obtaining the measurement result $\tilde{q}_1$ is

$$P(\tilde{q}_1) = \mathrm{Tr}_s[\hat{Y}\hat{\rho}_0\hat{Y}^\dagger] = \int dq\, {}_s\langle q|\hat{Y}\hat{\rho}_0\hat{Y}^\dagger|q\rangle_s \ , \tag{1}$$

which can be generally written as

$$P(\tilde{q}_1) \equiv \int dq\ \delta(q - \tilde{q}_1)P(q) \ . \tag{2}$$

Note that the statistics of the results of a measurement of an ensemble, where a single measurement is performed on each system in an ensemble of identical systems, would give the probability density $P(\tilde{q}_1)$. After the measurement, the system is described by the density operator

$$\hat{\rho} = P(\tilde{q}_1)^{-1}\hat{Y}\hat{\rho}_0\hat{Y}^\dagger \ . \tag{3}$$

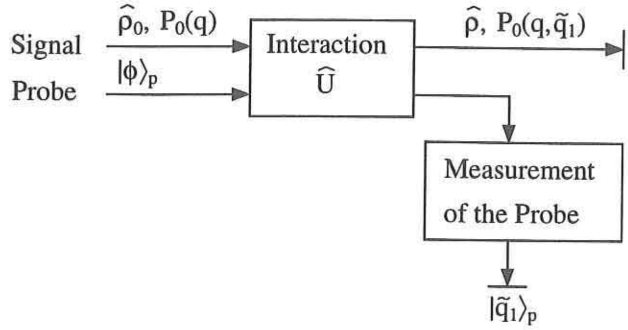

FIG. 1. The measurement of a quantum system, i.e., a signal, is composed of three 'stages: Preparation of a probe system, interaction of the probe with the signal, and measurement (which induces collapse) of the probe. In general, the probe and the signal are entangled after their interaction, and the collapse of the probe changes the wavefunction of the signal in accordance with the measurement result.

From Eqs. (2) and (3), the corresponding probability density of $\hat{q}$ is

$$P(q,\tilde{q}_1) = {}_s\langle q|\hat{\rho}|q\rangle_s = P(\tilde{q}_1)^{-1}\,\delta(q-\tilde{q}_1)\,P(q) \ . \tag{4}$$

Note that the probability density of $\hat{q}$ after a measurement is performed, $P(q,\tilde{q}_1)$, depends on the result of the measurement, $\tilde{q}_1$. The measurement process, therefore, modifies the wavefunction of the measured system in accordance with the measurement result. The next measurement is a precise measurement of $\hat{q}$ which results with $\tilde{q}_2$. The conditional probability to obtain $\tilde{q}_2$ in this measurement is

$$P(\tilde{q}_2|\tilde{q}_1) = \int dq\,\delta(q-\tilde{q}_2)\,P(q,\tilde{q}_1) \ . \tag{5}$$

Now, consider the statistics of the two measurement results, $\tilde{q}_1$ and $\tilde{q}_2$. Both measurement results can be used to estimate the center position of the probability density $P(\tilde{q}_1)$, $\langle\tilde{q}_1\rangle = \int d\tilde{q}_1\,P(\tilde{q}_1)\,\tilde{q}_1$, since

$$\langle\tilde{q}_2\rangle = \int d\tilde{q}_1\,P(\tilde{q}_1)\int d\tilde{q}_2\,P(\tilde{q}_2|\tilde{q}_1)\,\tilde{q}_2 = \langle\tilde{q}_1\rangle \ . \tag{6}$$

However, the width of the probability density $P(\tilde{q}_1)$, i.e., $\langle\Delta\tilde{q}_1^2\rangle = \langle\tilde{q}_1^2\rangle - \langle\tilde{q}_1\rangle^2$, where $\langle\tilde{q}_1^2\rangle = \int d\tilde{q}_1\,P(\tilde{q}_1)\,\tilde{q}_1^2$, cannot be estimated, because $\langle\tilde{q}_1\rangle^2$ cannot be estimated:

$$\langle\tilde{q}_2^2\rangle = \int d\tilde{q}_1\,P(\tilde{q}_1)\int d\tilde{q}_2\,P(\tilde{q}_2|\tilde{q}_1)\,\tilde{q}_2^2 = \langle\tilde{q}_1^2\rangle \ , \tag{7}$$

$$\langle\tilde{q}_1\tilde{q}_2\rangle = \int d\tilde{q}_1\,P(\tilde{q}_1)\,\tilde{q}_1\int d\tilde{q}_2\,P(\tilde{q}_2|\tilde{q}_1)\,\tilde{q}_2 = \langle\tilde{q}_1^2\rangle \ . \tag{8}$$

If $\tilde{q}_1$ and $\tilde{q}_2$ were independent results, obtained from two different quantum systems, which belong to the same ensemble and are, therefore, initially in the same quantum state, their correlation would be $\langle \tilde{q}_1 \tilde{q}_2 \rangle = \langle \tilde{q}_1 \rangle \langle \tilde{q}_2 \rangle = \langle \tilde{q}_1 \rangle^2$. This correlation, then, would provide the missing information about $\langle \tilde{q}_1 \rangle^2$, and $\langle \Delta \tilde{q}_1^2 \rangle$ could be estimated using both measurement results. In our case the conditional probability density to obtain the second measurement result, $\tilde{q}_2$, depends on the first measurement result, $\tilde{q}_1$. Therefore, the correlation of the two measurement results, which are taken from the same quantum system, does not give information about $\langle \tilde{q}_1 \rangle^2$, rather it gives $\langle \tilde{q}_1^2 \rangle$.

This is the main difference between the information provided by a measurement of an ensemble and the information provided by a series of measurements of a single system: While a measurement of an ensemble gives the probability density $P(\tilde{q}_1)$, a series of measurements of a single system does not, since the wavefunction of the measured system changes each time a measurement is performed in accordance with the measurement result, as a direct consequence of the generalized projection postulate. This change cannot be corrected for using unitary time evolution in between the two measurements without a-priori knowledge of the initial wavefunction of the measured system, just as one cannot devise a "protective measurement" for a system without a-priori knowledge of the state of the system (1,2). It may be possible to correct for that change using a measurement process, as in the case of the "reversible measurements" (3). However, the probability that the measurement process be reversed successfully is finite, and it is not certain that the wavefunction of the measured system will return to its original unknown state. Taking into account this finite success probability, the statistics of the results of a series of successfully reversed measurements of a single system are independent of the initial state of the measured system (4), and cannot be used to infer $P(\tilde{q}_1)$ which obviously depends on the initial state of the system.

The statistics of the results of a quantum measurement of the observable $\hat{q}$, as described in Eqs. (1)-(3), performed on an ensemble of systems, would always give the probability density $P(\tilde{q}_1)$. Yet, in order for this measurement of an ensemble to be considered as a determination of the wavefunction, one should be able to use $P(\tilde{q}_1)$ to infer $P_0(q) = {}_s\langle q|\hat{\rho}_0|q\rangle_s$, the initial probability density of $\hat{q}$ which is associated with the wavefunction of the measured system. This requires that the measurement process, i.e., the initial state of the probe, $|\phi\rangle_p$, the interaction between the probe and the measured system, $\hat{U}$, and the observable of the probe which measurement collapses the probe to any one of the states $|\tilde{q}_1\rangle_p$, would be chosen carefully. For example, one may choose to use a back-action evading (BAE) measurement process, in which the unitary time evolution operator, which describes the interaction of the signal and the probe, $\hat{U}$, is required to commute with the measured observable, $\hat{q}$,

$$[\hat{U}, \hat{q}] = 0 \ . \tag{9}$$

Indeed, the results of a BAE measurement of $\hat{q}$ performed on an ensemble would allow the inference of $P_0(q)$.

In our general model, therefore, we consider a series of alternating BAE measurements of the two conjugate observables $\hat{q}$ and $\hat{p}$, performed on a single quantum system. The statistics of the $\hat{q}$ measurement results are expected to give information about the initial probability density of $\hat{q}$, $P_0(q)$, i.e., estimates of the initial center position of this probability density (or the expectation value of $\hat{q}$), $\langle q_0\rangle = \int dq\, P_0(q)\, q$, and the initial width of the probability density (or the uncertainty associated with $\hat{q}$), $\langle \Delta q_0^2\rangle = \langle q_0^2\rangle - \langle q_0\rangle^2$, where $\langle q_0^2\rangle = \int dq\, P_0(q)\, q^2$. In the same way, the statistics of the $\hat{p}$ measurement results are expected to give information about $P_0(p) = {}_s\langle p|\hat{\rho}_0|p\rangle_s$. Note that this model applies to the case of repeated measurements of the observable $\hat{q}\cos\theta + \hat{p}\sin\theta$, for all $\theta \in [0, 2\pi]$. Indeed, one needs, at least, information about the probability densities of all of these observables in order to reconstruct the wavefunction of the measured system.

The first measurement is a measurement of $\hat{q}$. From Eqs. (1) and (9), the probability of obtaining the measurement result $\tilde{q}_1$ in this measurement is

$$P(\tilde{q}_1) = \int dq\, X(q, \tilde{q}_1)\, P_0(q) \quad , \tag{10}$$

where $X(q, \tilde{q}_1) = {}_s\langle q|\hat{X}(\hat{q}, \tilde{q}_1)|q\rangle_s$ is the probability for the probe to undergo a transition from the state $|\phi\rangle_p$ to the state $|\tilde{q}_1\rangle_p$ when the signal is in the state $|q\rangle_s$, and $\hat{X}(\hat{q}, \tilde{q}_1) = \hat{Y}^\dagger\hat{Y}$ is the generalized projection operator. Note that $X(q, \tilde{q}_1)$ depends only on the different aspects of the measurement process, while $P_0(q)$ depends only on the initial state of the measured system.

We assume that the measurement processes satisfy the following three conditions. Note that while these conditions are reasonable, they are not necessary for the following proof to be valid. First, the transition probability of the probe is required to be normalized over all possible final states of the probe,

$$\int d\tilde{q}_1\, X(q, \tilde{q}_1) = 1 \quad . \tag{11}$$

As the inferred value of $\hat{q}$, $\tilde{q}_1$ should equal, on average, the center position of the probability density of $\hat{q}$, $\langle\tilde{q}_1\rangle = \langle q_0\rangle$. This leads to the second condition,

$$\int d\tilde{q}_1\, X(q, \tilde{q}_1)\, \tilde{q}_1 = q \quad . \tag{12}$$

The signal and the probe should be independent of each other. Therefore, the probability error associated with the measurement result, $\tilde{q}_1$, should equal the sum of the measurement error, Δ_m^2, and the intrinsic uncertainty, due to the initial width of the probability density, $\langle\Delta\tilde{q}_1^2\rangle = \langle\tilde{q}_1^2\rangle - \langle\tilde{q}_1\rangle^2 = \langle\Delta q_0^2\rangle + \Delta_m^2$. From this we obtain the third condition,

$$\int d\tilde{q}_1\, X(q, \tilde{q}_1)\, \tilde{q}_1^2 = q^2 + \Delta_m^2 \quad . \tag{13}$$

Using Eqs. (3), (9) and (10), the probability density of $\hat{q}$ after the first measurement is

$$P(q,\tilde{q}_1) = P(\tilde{q}_1)^{-1}\,X(q,\tilde{q}_1)\,P_0(q) \ . \tag{14}$$

As before, $P(q,\tilde{q}_1)$ depends on $\tilde{q}_1$.

Next, the conjugate observable $\hat{p}$ is measured. According to the Heisenberg uncertainty relations, it is impossible to avoid a change in the probability density of $\hat{q}$ due to this measurement (unless some a-priori information about the state of the system before the measurement is given). We assume, though, that this change is the minimum change possible, in order to investigate the fundamental restrictions imposed on the measurement of the wavefunction of a single quantum system. In this case, the probability density of $\hat{q}$ changes from $P(q,\tilde{q}_1)$ to $P_1(q,\tilde{q}_1)$ as follows: The center position is unchanged,

$$\int dq\, P_1(q,\tilde{q}_1)\, q = \int dq\, P(q,\tilde{q}_1)\, q \ , \tag{15}$$

but the width increases due to the back-action noise, Δ_b^2,

$$\int dq\, P_1(q,\tilde{q}_1)\, q^2 = \int dq\, P(q,\tilde{q}_1)\, q^2 + \Delta_b^2 \ . \tag{16}$$

Now $\hat{q}$ is measured for the second time. Following the treatment of the first measurement of $\hat{q}$ in Eqs. (10)-(13), the conditional probability to obtain $\tilde{q}_2$ in this measurement, after $\tilde{q}_1$ is obtained in the previous measurement, is

$$P(\tilde{q}_2|\tilde{q}_1) = \int dq\, X(q,\tilde{q}_2)\, P_1(q,\tilde{q}_1) \ . \tag{17}$$

Consider the statistics of the results of the two measurements of $\hat{q}$. Each of the measurement results, $\tilde{q}_1$ or $\tilde{q}_2$, can estimate the initial center position, $\langle q_0 \rangle$. Indeed, as can be seen from Eq. (12), this is one of the three conditions the measurements are assumed to satisfy,

$$\langle \tilde{q}_1 \rangle = \langle q_0 \rangle \ , \tag{18}$$

$$\langle \tilde{q}_2 \rangle = \langle q_0 \rangle \ . \tag{19}$$

As was mentioned before, this assumption is not necessary for our general proof to be valid. We make this assumption because, obviously, it is possible to estimate the expectation values of different observables from the results of measurements performed on a single quantum system. Therefore, it is not the lack of information about these expactation values (or the center positions of the corresponding probability densities) which inhibits the measurement of the wavefunction of a single quantum system. The second order moment $\langle q_0^2 \rangle$ can also be estimated using either $\tilde{q}_1$ or $\tilde{q}_2$, since

$$\langle \tilde{q}_1^2 \rangle = \langle q_0^2 \rangle + \Delta_m^2 \ , \tag{20}$$

$$\langle \tilde{q}_2^2 \rangle = \langle q_0^2 \rangle + \Delta_m^2 + \Delta_b^2 \ , \tag{21}$$

where the measurement error, Δ_m^2, and the back-action noise, Δ_b^2, are parameters of the measurement process, independent of the state of the measured system, and therefore known to us. One cannot, however, estimate the initial width of the probability density of $\hat{q}$, $\langle \Delta q_0^2 \rangle$, using a single measurement result, because a single measurement result does not contain information about $\langle q_0 \rangle^2$. As before, if $\tilde{q}_1$ and $\tilde{q}_2$ were each obtained from one of two independent quantum systems, which belong to the same ensemble, their correlation would provide the missing information, $\langle \tilde{q}_1 \tilde{q}_2 \rangle = \langle q_0 \rangle^2$, and $\langle \Delta q_0^2 \rangle$ could be estimated using both measurement results. In our case the second measurement result, $\tilde{q}_2$, depends on the first, $\tilde{q}_1$, due to the change imposed on the wavefunction of the measured system by the measurement process. Therefore, the correlation of the two measurement results, which are taken from the same quantum system, does not give information about $\langle q_0 \rangle^2$, rather it gives

$$\langle \tilde{q}_1 \tilde{q}_2 \rangle = \langle q_0^2 \rangle \ . \tag{22}$$

In fact, in order for the measurement results $\tilde{q}_1$ and $\tilde{q}_2$ to give an estimate of $\langle q_0 \rangle^2$, the transition probabilities $X(q, \tilde{q}_1)$ and $X(q, \tilde{q}_2)$ should depend on $\langle q_0 \rangle$. This is impossible, since $X(q, \tilde{q}_1)$ and $X(q, \tilde{q}_2)$ are independent of the state of the measured system, as can be seen from Eq. (10). The conditions of Eqs. (11)-(13) are, therefore, not necessary for our conclusion to be valid.

This treatment can be easily extended to include as many measurements of $\hat{q}$ as we want, by way of mathematical induction. For the k-th measurement result, we obtain

$$\langle \tilde{q}_k \rangle = \langle q_0 \rangle \ , \tag{23}$$

$$\langle \tilde{q}_k^2 \rangle = \langle q_0^2 \rangle + \Delta_m^2 + (k-1)\Delta_b^2 \ , \tag{24}$$

$$\langle \tilde{q}_k \tilde{q}_l \rangle = \langle q_0^2 \rangle + (k-1)\Delta_b^2 \ , \quad \text{for all } k < l \ . \tag{25}$$

Regardless of the number of measurements of $\hat{q}$ performed on the single system, the information about $\langle q_0 \rangle^2$ is always missing, and $\langle \Delta q_0^2 \rangle$ cannot be estimated. The correlation of any two measurement results, which are taken from a single quantum system, gives information about $\langle q_0^2 \rangle$, where the same information can, actually, be obtained from the result of a single measurement.

A similar treatment can be used to analyze the results of the measurements of $\hat{p}$. Always the conclusion is the same: While it is possible to estimate the initial center positions of $P_0(q)$ and $P_0(p)$ with a linear function of the corresponding measurement results, no quadratic function of the measurement results can estimate the initial widths of $P_0(q)$ and $P_0(p)$. Since no information about the widths of the probability densities is obtained, the process of repeated measurements is equivalent to a measurement of the observables $\hat{q}$ and $\hat{p}$, and cannot be considered as a measurement of the wavefunction. The same is true for all processes of repeated quantum measurements of the arbitrary observable $\hat{q}\cos\theta + \hat{p}\sin\theta$, regardless of the strength of the measurements.

This analysis shows that the wavefunction of a single system cannot be measured by a series of quantum measurements. Each time a measurement is performed, the wavefunction changes in accordance with the measurement result. Therefore, the statistics of the measurement results contain no information about the initial uncertainties of the measured observables, i.e., the initial widths of the corresponding probability densities. The change of the wavefunction due to the measurement process is a direct consequence of the projection postulate. It is shown, then, that the projection postulate limits the quantum wavefunction to have a statistical (or epistemological) meaning only.

In this proof, the time evolution of the quantum system in between measurements is neglected, while we concentrate on the relevant changes in the wavefunction due to the measurement process. One may assume that the measurements are performed one immediately after the other, with no time delay in between consecutive measurements. Alternatively, one may assume that the measured observables, $\hat{q}$ and $\hat{p}$, are QND observables, and therefore their probability densities do not change due to the free time evolution of the system.

We now illustrate these general considerations with two examples.

REPEATED PHOTON-NUMBER QND MEASUREMENTS OF A SINGLE SQUEEZED STATE OF LIGHT: SATURATED QUANTUM BROWNIAN MOTION AND CONTINUOUS COLLAPSE OF THE WAVEFUNCTION

The first example is the case of a series of photon-number QND measurements performed on a single squeezed state of light (8). Each time a measurement is performed, the signal state is correlated to a probe state in an optical Kerr medium (Fig. 2). The probe is prepared in a squeezed state, $|\alpha, r\rangle_p$, with the excitation $|\alpha|^2$ and the squeezing parameter r. This interaction process is described by the unitary operator $\hat{U}(\hat{n}_s) = \exp(i\mu\,\hat{n}_s\hat{n}_p)$, where $\hat{n}_s$ and $\hat{n}_p$ are the photon-number operators of the signal and the probe respectively, and μ is the coupling strength (17). The photon-number of the signal, $\hat{n}_s$, modulates the refractive index of the Kerr medium, and shifts the phase of the probe, $\Delta\hat{\phi}_p = \mu\hat{n}_s$. Then, the (slowly varying) second-quadrature amplitude of the probe, $\hat{a}_{2,p}$, is measured precisely by a homodyne detection. If the initial phase of the probe is zero, and the phase shift is small, due to a weak coupling, $\mu \ll 1$, then $\hat{a}_{2,p}$ is approximately linear with the phase shift of the probe, and with the photon-number of the signal, $\hat{a}_{2,p} \cong |\alpha|\mu\hat{n}_s$. In this case, the measurement result, α_2, gives the inferred photon-number of the signal, $\tilde{n} \cong \alpha_2/(|\alpha|\mu)$. The corresponding measurement error is $\Delta_m^2 = \langle\Delta\hat{a}_{2,p}^2\rangle/(|\alpha|\mu)^2$, where $\langle\Delta\hat{a}_{2,p}^2\rangle = e^{-2r}/4$ is the initial uncertainty of the second-quadrature amplitude of the probe. Note that the limit imposed on the coupling strength, $\mu \ll 1$, does not limit the strength of the measurement, $1/\Delta_m^2$, since $\langle\Delta\hat{a}_{2,p}^2\rangle$ is not limited. Also note that the back-action noise, which is imposed on the phase of the signal by the photon-

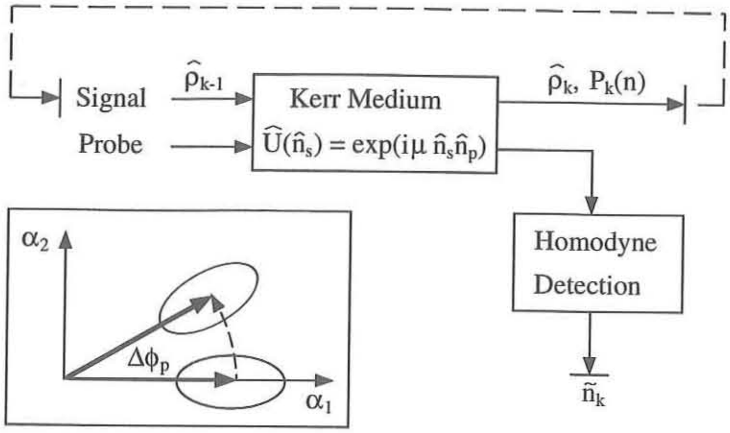

FIG. 2. Repeated weak photon-number QND measurements: In each measurement the signal is correlated to a new probe in an optical Kerr medium. The inset shows the shift in the phase of the probe due to this correlation. The second-quadrature amplitude of the output probe is measured precisely by a homodyne detection. The inferred photon-number is obtained from the result of this measurement. The signal output is then measured again.

number of the probe, $\Delta\hat{\phi}_s = \mu\hat{n}_p$, does not influence the probability density of the photon-number of the signal, since $\hat{n}_s$ is a QND observable.

The initial probability density of the second-quadrature amplitude of the probe, the probe being in a squeezed state with a zero phase, is a Gaussian, centered at zero, with the variance $\langle\Delta\hat{a}^2_{2,p}\rangle$. Therefore, the probability-amplitude operator which describes this measurement process, $\hat{Y}(\hat{n}_s, \tilde{n}) = {}_p\langle\tilde{n}|\hat{U}(\hat{n}_s)|\alpha, r\rangle_p$, corresponds to a Gaussian transition probability (18),

$$X(n, \tilde{n}) = N[\tilde{n}, n, \Delta^2_m] \ . \tag{26}$$

Let us assume that the initial photon-number probability density of the signal is also a Gaussian,

$$P_0(n) = N[n, n_0, \Delta^2_0] \ . \tag{27}$$

Physically, the photon-number probability density of a squeezed state is a discrete sub- or super-Poissonian probability density, with $n \geq 0$. If the initial excitation of the signal is large, i.e., $n_0 \gg 1$, this Gaussian approximation is valid. The following model, therefore, describes a measurement process in which both the signal and the probe have Gaussian probability densities. Many other physical situations are described in the same way. One such example is the QND measurement of one of the quadrature amplitudes of a squeezed state of light, using a non-degenerate parametric amplification.

The same measurement procedure is repeated k times. Each time, the measurement is performed on the output signal of the previous measurement, using a new probe state. We get a series of second-quadrature amplitude readouts, which corresponds to a series of inferred photon-number values, $(\tilde{n}_1, \tilde{n}_2, ..., \tilde{n}_k)$. It is the statistics of $(\tilde{n}_1, \tilde{n}_2, ..., \tilde{n}_k)$, in the limit of weak measurements, which are expected to give the initial photon-number probability density of the signal, $P_0(n)$. The total probability-amplitude operator which describes the whole process of k repeated QND measurements is

$$\hat{Z}_k(\hat{n}_s) = \hat{Y}(\hat{n}_s, \tilde{n}_k) ... \hat{Y}(\hat{n}_s, \tilde{n}_2) \hat{Y}(\hat{n}_s, \tilde{n}_1) \ . \tag{28}$$

Note that $\hat{Z}_k(\hat{n}_s)$, is symmetric in the measurement results, $(\tilde{n}_1, \tilde{n}_2, ..., \tilde{n}_k)$, i.e., it is independent of the order in which the results are obtained. This is because the different probability-amplitude operators commute with each other, i.e., $[\hat{Y}(\hat{n}_s, \tilde{n}_k), \hat{Y}(\hat{n}_s, \tilde{n}_l)] = 0$, for all k and l. Therefore, the probability of obtaining these results, $P(\tilde{n}_1, \tilde{n}_2, ..., \tilde{n}_k)$, and the photon-number probability density after these results are obtained, $P_k(n)$, are both symmetric in $(\tilde{n}_1, \tilde{n}_2, ..., \tilde{n}_k)$. The probability of obtaining $\tilde{n}_2$ in the second measurement depends on the result of the first measurement, $\tilde{n}_1$. Yet, the process of measuring $\tilde{n}_1$ first and $\tilde{n}_2$ second has exactly the same probability as the process in which $\tilde{n}_2$ is measured first and $\tilde{n}_1$ is measured second. Also, the changes that these two processes impose on the wavefunction are exactly the same. Since the wavefunction of the system changes slightly from one measurement to another, the above observation, that the total probability-amplitude operator, $\hat{Z}_k(\hat{n}_s)$, is independent of the order of the measurement results, already suggests that no information about the width of the photon-number probability density is contained in the statistics of the readouts, $P(\tilde{n}_1, \tilde{n}_2, ..., \tilde{n}_k)$.

To confirm this we first consider the case of a single measurement perfomed on a single squeezed state. Then, we examine the conventional measurement of the wavefunction, the case of a measurement of an ensemble of squeezed states. Finally, we analyze the case of repeated measurements performed on a single state. We show that the statistics of the measurement results contain no information about the width of the photon-number probability density, due to the changes, which are imposed on the measured state by the process of repeated measurements. We also discuss the consistency of this result with Holevo's theorem of the quantum channel capacity.

A Measurement of a Single Squeezed State

Consider the case of a single photon-number QND measurement performed on a single squeezed state of light. The probability of measuring $\tilde{n}_1$, is, according to Eqs. (10), (26) and (27),

$$P(\tilde{n}_1) = N[\tilde{n}_1, n_0, \Delta_0^2 + \Delta_m^2] \ . \tag{29}$$

As can be seen, $\tilde{n}_1$ is an estimate of the initial center position of the photon-number probability density, n_0, with the estimate error being $\Delta_0^2 + \Delta_m^2$. Obviously, the initial width of the probability density, Δ_0^2, cannot be estimated using this single measurement result. From Eqs. (14), (26) and (27), the photon-number probability density of the signal after the measurement is

$$P_1(n) = N[n, n_0^1, \Delta_1^2] \ , \tag{30}$$

$$n_0^1 = \Delta_1^2 \, (n_0/\Delta_0^2 + \tilde{n}_1/\Delta_m^2) \ , \tag{31}$$

$$\Delta_1^2 = (1/\Delta_0^2 + 1/\Delta_m^2)^{-1} \ . \tag{32}$$

Due to this measurement, the center of the photon-number probability density, is shifted from n_0 to n_0^1, toward the measurement result, $\tilde{n}_1$. The width of the probability density narrows from Δ_0^2 to Δ_1^2. Note that the larger the initial width, Δ_0^2, the larger the relative reduction in the width, $(\Delta_0^2 - \Delta_1^2)/\Delta_0^2 = \Delta_0^2/(\Delta_0^2 + \Delta_m^2)$, and, for a given measurement result, $\tilde{n}_1$, the more significant the shift in the center position. The changes due to the measurement are more dramatic for probability densities which are initially wide, than for those which are initially narrow.

If the measurement is weak, $\Delta_m^2 \gg \Delta_0^2$, both these shift and narrowing are very small. In this case, the squeezed state could be measured many times before its photon-number probability density would change appericiably.

The Conventional Measurement of the Wavefunction: A Measurement of an Ensemble of Squeezed States

Usually, the wavefunction is measured on an ensemble of systems, all prepared in the same initial state. More specifically, the wavefunction is obtained from the statistics of the results of these measurements. Therefore, before investigating the case of repeated QND measurements performed on a single squeezed state, we analyze the case of one measurement performed on each state in an ensemble of k squeezed states. In this case, each measurement is independent of the others. The probability of obtaining the inferred photon-number values, $(\tilde{n}_1, \tilde{n}_2, ..., \tilde{n}_k)$, is, obviously, independent of their order, and therefore $P(\tilde{n}_1, \tilde{n}_2, ..., \tilde{n}_k) = \prod_{i=1}^{k} P(\tilde{n}_i)$.

It is well known (19) that the statistics of the results of the measurements in this case are analyzed by the estimated center position (or expectation value), $\overline{n}$, and the estimated width (or uncertainty), $\overline{\Delta n^2}$,

$$\overline{n} = \frac{1}{k} \sum_{i=1}^{k} \tilde{n}_i \ , \tag{33}$$

$$\overline{\Delta n^2} = \frac{1}{k-1} \sum_{i=1}^{k} (\tilde{n}_i - \overline{n})^2 \ , \tag{34}$$

in which all measurement results have the same weight. From Eq. (29), the probability that the measurements performed on the ensemble would result with $(\tilde{n}_1, \tilde{n}_2, ..., \tilde{n}_k)$, in terms of $\overline{n}$ and $\overline{\Delta n^2}$, is

$$\prod_{i=1}^{k} P(\tilde{n}_i) d\tilde{n}_i = [P(\overline{n})\, d\overline{n}]\, [P(S)\, dS]\, d\Omega_{k-1} \ , \tag{35}$$

$$S \equiv [(k-1)/(\Delta_0^2 + \Delta_m^2)]\, \overline{\Delta n^2} \ , \tag{36}$$

where $d\Omega_{k-1}$ is a normalized infinitesimal element of the solid angle in dimension $(k-1)$, $\int d\Omega_{k-1} = 1$.

The probability distribution of the estimated center position is

$$P(\overline{n}) = N[\overline{n}, n_0, (\Delta_m^2 + \Delta_0^2)/k] \ . \tag{37}$$

On average, $\overline{n}$ equals n_0, and therefore, $\overline{n}$ is indeed a statistical estimate of the center position of the initial photon-number probability density. The variance of $\overline{n}$ is inversely proportional to the number of measurements, k. Therefore, the probability error associated with this estimate decreases as the number of measurement results increases. The probability distribution of S is a chi-squared distribution (20),

$$P(S) = \chi^2[S, (k-1)] \ , \tag{38}$$

and the distribution of the estimated width, $\overline{\Delta n^2}$, is centered at $\Delta_0^2 + \Delta_m^2$, with the variance $2(\Delta_0^2 + \Delta_m^2)^2/(k-1)$. As k increases, the probability error for $\overline{\Delta n^2}$ to read $\Delta_0^2 + \Delta_m^2$ decreases. We can conclude that, by measuring an ensemble of squeezed states, all with the same initial wavefunction, both the center and width of the initial photon-number probability density associated with this wavefunction can be estimated statistically. The information about the width of the photon-number probability density, which is not available from a single measurement of a single squeezed state, makes the measurement of an ensemble of squeezed states a measurement of the wavefunction, as opposed to a measurement of the photon-number alone.

The Changes in a Single Squeezed State Due to Repeated Photon-Number QND Measurements

Next, let us consider the changes in a single squeezed state in the process of k repeated measurements. From Eqs. (10), (14) and (26)-(28), we obtain that the final photon-number probability density after k repeated measurements, which result in $(\tilde{n}_1, \tilde{n}_2, ..., \tilde{n}_k)$, is

$$P_k(n) = N[n, n_0^k, \Delta_k^2] \ , \tag{39}$$

$$n_0^k = \Delta_k^2 \, (n_0/\Delta_0^2 + \sum_{i=1}^{k} \tilde{n}_i/\Delta_m^2) \ , \tag{40}$$

$$\Delta_k^2 = (1/\Delta_0^2 + k/\Delta_m^2)^{-1} \ . \tag{41}$$

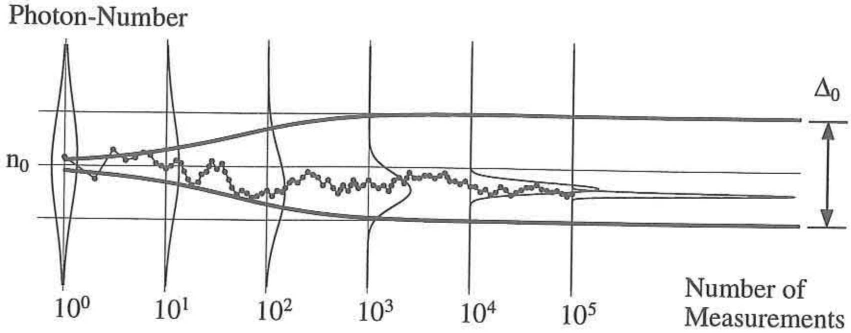

FIG. 3. The quantum Brownian motion and the continuous collapse of the photon-number probability density of a single squeezed state, in the process of repeated weak photon-number QND measurements. The initial probability density, shown on the left, is centered at n_0 with the width Δ_0. The thick lines describe the statistical diffusion of the center position of this probability, which reaches the initial width of the probability density. The explicit drawings of the probability density demonstrate its continuous collapse. Note that the effect of earlier measurements on the wavefunction is more dramatic than the effect of later measurements.

As was noted before, $P_k(n)$ is symmetric in $(\tilde{n}_1, \tilde{n}_2, ..., \tilde{n}_k)$. Also note, by comparing Eqs. (39)-(41) with Eqs. (30)-(32), that the total change in the wavefunction due to k repeated measurements of strength $1/\Delta_m^2$ is exactly the same as the change due to a single measurement of a strength k/Δ_m^2, which results in $\tilde{n}_1 = \overline{n}$. After each measurement, the width of the photon-number probability density decreases (continuous collapse). The center of this probability density takes a step in a random walk (quantum Brownian motion), which depends on the random result of the measurement (Fig. 3).

The probability distribution which statistically describes the diffusion of the center position of the photon-number probability density after k measurements, n_0^k, is

$$P(n_0^k) = N[n_0^k, n_0, (k/\Delta_m^2)\Delta_0^2\Delta_k^2] \ . \tag{42}$$

On average, the center position is always at the initial center position, n_0. However, the probability of finding the center farther away from n_0 increases as the number of measurements increases. As long as the total strength of the measurements is small, $k/\Delta_m^2 \ll 1/\Delta_0^2$, the variance of n_0^k increases linearly with the number of measurements, $(k/\Delta_m^2)\Delta_0^2\Delta_k^2 \approx Dk$. In this regime the movement of the center position is a quantum Brownian motion with a constant diffusion coefficient $D = \Delta_0^4/\Delta_m^2$. Here the time scale is replaced by the discrete scale of the number of measurements. As the photon-number probability density narrows, the average step size of this quantum Brownian motion decreases. The statistical variance of n_0^k saturates, and equals the initial width of the photon-number probability density (the initial uncertainty associated with the photon-number of the squeezed state), Δ_0^2. At the same time, the squeezed state is reduced to a photon-number eigenstate. The measured state, therefore, undergoes quantum Brownian motion, which is saturated due to its continuous collapse.

The Statistics of the Results of Repeated Measurements Performed on a Single Squeezed State

Analyzing the statistics of the results of k repeated measurements performed on a single squeezed state, we use the same definitions for the estimates of the center position and width of the photon-number probability density as for the case of k measurements performed on an ensemble. These definitions, which appear in Eqs. (33) and (34), are symmetric in the measurement results, $(\tilde{n}_1, \tilde{n}_2, ..., \tilde{n}_k)$. In the case of k repeated measurements performed on a single squeezed state, both the final photon-number probability density of this state, $P_k(n)$, and the probability to obtain a specific series of results, $P(\tilde{n}_1, \tilde{n}_2, ..., \tilde{n}_k)$, are independent of the order in which these results are obtained. Therefore, it is natural to use the same $\overline{n}$ and $\overline{\Delta n^2}$ as before. In fact, $\overline{n}$ as defined in Eq. (33) can be shown to be the best estimate for the center position of the photon-number probability density in this case: As an estimate of the center position, $\overline{n}$ should be linear in the measurement results, $\overline{n} = \sum_{i=1}^{k} c_i \tilde{n}_i$. For $\overline{n}$ to be the best estimate of the center position, it should equal the center position, on average, $\langle \overline{n} \rangle = n_0$, and the associated estimate error should be minimized $d\langle \Delta \overline{n}^2 \rangle / dc_i = 0$. These requirements result with $c_i = 1/k$ for all i, and the definition of Eq. (33) is recovered.

From Eqs. (10) and (26)-(28), the probability of getting the measurement results $(\tilde{n}_1, \tilde{n}_2, ..., \tilde{n}_k)$ in the process of k repeated measurements performed on a single state is

$$P(\tilde{n}_1, \tilde{n}_2, ..., \tilde{n}_k) \prod_{i=1}^{k} d\tilde{n}_i = [P(\overline{n})\, d\,\overline{n}]\, [P(S)\, dS]\, d\Omega_{k-1} \quad , \tag{43}$$

$$S \equiv [(k-1)/\Delta_m^2]\, \overline{\Delta n^2} \quad . \tag{44}$$

Comparing Eqs. (36) and (44) we see that in this case, unlike the case of k measurements performed on an ensemble, S is independent of Δ_0^2.

The probability distribution of the estimated center position is

$$P(\overline{n}) = N[\overline{n}, n_0, \Delta_0^2 + \Delta_m^2/k] \quad . \tag{45}$$

The error associated with $\overline{n}$ as an estimate of the center position, n_0, decreases with an increased number of measurements, and, as $k \to \infty$, this error reaches its minimum value which equals the initial uncertainty in the photon-number of the squeezed state, Δ_0^2. Therefore, the estimated center position has the same probability error in both cases of an infinite number of repeated weak measurements and one precise measurement. In fact, comparing Eq. (45) with Eq. (29), we see that the probabilities to infer a certain $\overline{n}$, by k consecutive measurements of strength $1/\Delta_m^2$, and by one measurement of a strength k/Δ_m^2, are equal. The probability distribution of S is, again, a chi-squared distribution,

$$P(S) = \chi^2[S, (k-1)] \ . \tag{46}$$

However, $P(S)$ is now independent of Δ_0^2, and, therefore, the estimated width, $\overline{\Delta n^2}$, is not a statistical measure of the initial width, Δ_0^2. Indeed, $\overline{\Delta n^2}$ equals Δ_m^2 on average, which means that $\overline{\Delta n^2}$ estimates the measurement error (which we, obviously, know already), with the estimate error being $2\Delta_m^4/(k-1)$. The statistics of the results of repeated weak QND measurements performed on a single squeezed state contain no information about the initial width of the photon-number probability density of this state. In contradiction with our expectations, these statistics do not infer the wavefunction of the single squeezed state.

Consistency with Holevo's Theorem of the Quantum Channel Capacity

The above conclusion, that the results of repeated photon-number measurements, performed on a single squeezed state of light, can be used to estimate the photon-number expectation value, but not the uncertainty associated with the photon-number, is consistent with the fundamental theorem in quantum communication theory, Holevo's theorem (21).

Every physical means of information transfer can be considered as a communication channel. When the information is tranmitted using quantum states, and detected using quantum measurements, this is a quantum communication channel. The quantum communication channel can be described schematically as follows: To transmit the information, the sender selects one pure quantum state, with the density operator $\hat{\rho}_i$, out of a set of possible states $\{\hat{\rho}_i\}$. The probability that the sender chooses this particular state is $P(\hat{\rho}_i)$. The receiver does not know which of the pure states the sender selects each time, but knows $P(\hat{\rho}_i)$, the probability distribution for these states to be selected. Therefore, the receiver views the states received in the quantum communication channel as mixed states, described by the density operator $\hat{\rho} = \sum_i P(\hat{\rho}_i)\,\hat{\rho}_i$. In order to maximize the information transferred in the quantum communication channel, the sender can choose between different sets of pure states, and different probability distributions of selecting a specific state. The receiver can choose between different quantum measurements. However, according to Holevo's theorem, the maximum information transfer in a quantum channel, or the channnel capacity, is limited, regardless of the choices made by the sender and the receiver. The maximum capacity for a quantum channel is $S_{max} = \text{Max}[S(\hat{\rho})]$, where $S(\hat{\rho}) = -\text{Tr}(\hat{\rho}\log_2\hat{\rho})$ is the quantum entropy of the density operator $\hat{\rho}$. $S(\hat{\rho})$ characterizes the number of distinguishable pure states in the set of states $\{\hat{\rho}_i\}$.

In the case of the bosonic channel, the maximium channel capacity can be realized by the number eigenstates channel (22,23), in which the set of input quantum states is the set of number (or energy) eigenstates, the probability distribution for the sender to choose one of these states is the so-called thermal distribution, $P(n_i) = \langle n \rangle^{n_i}/(\langle n \rangle + 1)^{n_i+1}$, where $\langle n \rangle$ is the average number, and the measurements, which are done by the receiver, are ideal number (or energy) measurements.

Now, if it were possible to measure the photon-number probability density of a single squeezed state, i.e., if it were possible to estimate both the expectation value and the uncertainty associated with the photon-number of a single state, one could consider encoding information on the photon-number uncertainty, and not only the photon-number expectation value. Unless some information about the photon-number expectation value is lost, when some information about the photon-number uncertainty is gained, the process of repeated weak measurements would make it possible to exceed the maximum capacity of the bosonic channel. This would be a violation of Holevo's theorem. As we have shown, a series of repeated weak QND measurements infers the center position of the photon-number probability density, or the photon-number expectation value, with the same probability error as a single strong measurement does. No information about the photon-number expectation value is lost. Therefore, in order to avoid a direct violation of Holevo's theorem, it is required that no information about the photon-number uncertainty could be gained. Indeed, this series of measurements cannot infer the photon-number uncertainty with a finite probability error, and distinguish between different states of equal photon-number expectation values and different photon-number uncertainties.

The Relation Between the Changes in the Wavefunction and the Statistics of the Measurement Results or Why is the Projection Postulate Responsible for the Inhibition of the Measurement of the Wavefunction?

In the process of repeated photon-number QND measurements, performed on a single squeezed state, the wavefunction of this state undergoes saturated quantum Brownian motion and continuous collapse. These changes in the wavefunction originate in the projection postulate. The statistics of the measurement results contain information about the expectation value of the photon-number. The information about the uncertainty associated with the photon-number, however, is cancelled out, due to the exact coordination between the quantum Brownian motion and the continuous collapse of the wavefunction. Therefore, this process of repeated measurements does not correspond to a measurement of the wavefunction of the single squeezed state, and this inhibition of the measurement of the wavefunction is a direct consequence of the projection postulate.

The first indication to the inhibition of the measurement of the wavefunction of the single squeezed state, in the above analysis of repeated photon-number QND measurements, is the symmetry of the statistics of the measurement results, $P(\tilde{n}_1, \tilde{n}_2, ..., \tilde{n}_k)$. Each time the squeezed state is measured, it is slightly changed. The results of the consecutive measurements are essentially collected from an ensemble of squeezed states with different photon-number probability densities. All these results have the same weight in $P(\tilde{n}_1, \tilde{n}_2, ..., \tilde{n}_k)$, and therefore their statistics are independent of the initial uncertainty in the photon-number. There is no natural way to assign different weights to the different results in the definitions of the estimates $\overline{n}$ and $\overline{\Delta n^2}$, since the changes in the wavefunction are also symmetric in $(\tilde{n}_1, \tilde{n}_2, ..., \tilde{n}_k)$, and we cannot overcome the symmetry of $P(\tilde{n}_1, \tilde{n}_2, ..., \tilde{n}_k)$.

Consider the case in which one is trying to distinguish between two squeezed states of large and small photon-number uncertainties, i.e., wide and narrow photon-number probability densities, both with the same initial photon-number expectation value, n_0, by repeatedly measuring both states. The first measurement result obtained from the wide state is more likely to be farther away from n_0 than the first result obtained from the narrow state. However, the shift toward the measurement result and the collapse due to the first measurement are more dramatic in the case of the wide state. Therefore, the probability of obtaining the second result in a certain distance from the first result can be the same for both states, regardless of the initial widths of their photon-number probability densities.

Note that while the continuous collapse of the wavefunction and the symmetry of the statistics of the measurement results are valid for all processes of repeated quantum measurements of the same observable, performed on a single quantum system, they do not necessarily hold in general, when more then a single observable is measured. The following example shows that even when the collapse of the wavefunction of the single system to an eigenstate of the measured observable is prevented, due to back-action noise imposed by measurements of the conjugate observable, and the statistics of the measurement results depend on the order in which these results are obtained, it is still impossible to measure the wavefunction of the single system.

ALTERNATING QND MEASUREMENTS OF THE TWO QUADRATURE AMPLITUDES OF A SINGLE SQUEEZED STATE OF LIGHT

The second example is the case of alternating QND measurements of the two (slowly varying) quadrature amplitudes of a squeezed state (Fig. 4), using

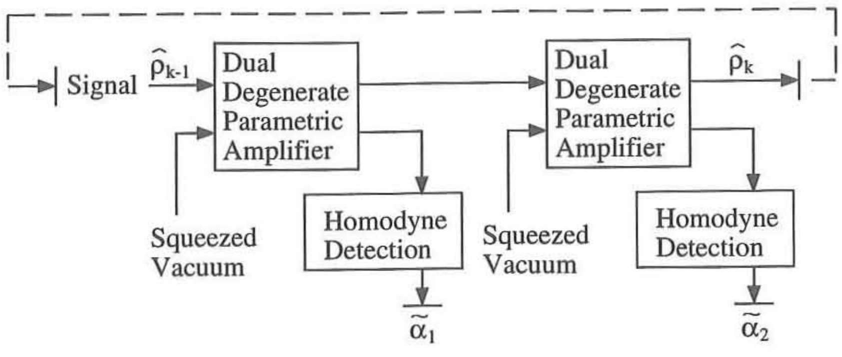

FIG. 4. Alternating QND measurements of the two (slowly varying) quadrature amplitudes: In each measurement, the signal is correlated to a new probe, which is in a squeezed vacuum state. Then, the first-quadrature (or the second-quadrature) amplitude of the probe is measured precisely using homodyne detection.

dual degenerate parametric amplification (13,14). Each time a measurement is taken, the signal squeezed state is correlated to a probe, which is prepared in a squeezed vacuum state. In the odd (even) measurements, the result of a measurement of the second-quadrature (first-quadrature) of the probe, $\hat{a}_{2,p}$ ($\hat{a}_{1,p}$), is used to infer the first-quadrature (second-quadrature) of the signal, $\hat{a}_{1,s}$ ($\hat{a}_{2,s}$). As in the previous example, both the signal and the probe have Gaussian probability densities, and the analysis of the photon-number QND measurement of Eqs. (30)-(32) can be used to describe the quadrature amplitude QND measurement.

The initial probability density of $\hat{a}_{1,s}$ is $P_0(\alpha) = N[\alpha, \alpha_0, \Delta_0^2]$. The probability density of $\hat{a}_{1,s}$ before the k-th measurement of $\hat{a}_{1,s}$, $P_{k-1}(\alpha)$, in terms of the probability density before the $(k-1)$-th measurement of $\hat{a}_{1,s}$, $P_{k-2}(\alpha)$, the measurement error associated with this measurement, Δ_m^2, and the back-action noise due to the $(k-1)$-th measurement of $\hat{a}_{2,s}$, Δ_b^2, is

$$P_{k-1}(\alpha) = N[\alpha, \alpha_0^{k-1}, \Delta_{k-1}^2] \ , \tag{47}$$

$$\alpha_0^{k-1} = (1/\Delta_{k-2}^2 + 1/\Delta_m^2)^{-1}(\alpha_0^{k-2}/\Delta_{k-2}^2 + \tilde{\alpha}_{k-1}/\Delta_m^2) \ , \tag{48}$$

$$\Delta_{k-1}^2 = (1/\Delta_{k-2}^2 + 1/\Delta_m^2)^{-1} + \Delta_b^2 \ , \tag{49}$$

where $\tilde{\alpha}_{k-1}$ is the result of the $(k-1)$-th measurement of $\hat{a}_{1,s}$. Note that for $\Delta_b^2 = 0$ this example is reduced to the case of repeated measurements of $\hat{a}_{1,s}$, which is mathematically equivalent to the first example of repeated photon-number measurements.

Consider the changes in the probability density of $\hat{a}_{1,s}$. Each measurement of the conjugate observable, $\hat{a}_{2,s}$, increases the width of this probability density by Δ_b^2, the back-action noise. The consecutive measurement of $\hat{a}_{1,s}$ narrows this probability density in proportion to its current width. Therefore, if the measurement of $\hat{a}_{2,s}$ increases the width of the probability density significantly, then the consecutive measurement of $\hat{a}_{1,s}$ would reduce it significantly. However, if the change in the width due to the measurement of $\hat{a}_{2,s}$ is insignificant, then the change due to the consecutive measurement of $\hat{a}_{1,s}$ is also relatively insignificant. The squeezed state is, obviously, prevented from collapsing to an eigenstate of $\hat{a}_{1,s}$ (or $\hat{a}_{2,s}$), but the narrowing and widening of the probability density of $\hat{a}_{1,s}$ (or $\hat{a}_{2,s}$) due to the alternating measurements of $\hat{a}_{1,s}$ and $\hat{a}_{2,s}$ would, eventually, balance, to keep the width of this probability density the same each time $\hat{a}_{1,s}$ (or $\hat{a}_{2,s}$) is measured, i.e., $\Delta_{k-1}^2 = \Delta_k^2$. In this limit, as can be seen from Eq. (49), the widths of the probability densities of $\hat{a}_{1,s}$ and $\hat{a}_{2,s}$ are solely determined by the relative strengths of the $\hat{a}_{1,s}$ and $\hat{a}_{2,s}$ measurements. Note that the back-action noise, Δ_b^2, that a measurement of $\hat{a}_{2,s}$ imposes on the probability density of $\hat{a}_{1,s}$, is determined by the strength of the $\hat{a}_{2,s}$ measurement. If the measurements of $\hat{a}_{1,s}$ and $\hat{a}_{2,s}$ have equal strengths, for instance, than the noise distribution of the squeezed state would reach that of a coherent state. After reaching this limit, the squeezed state would diffuse freely (quantum Brownian motion), while preserving its noise distribution, due to the process of repeated measurements.

The conditional probability to obtain $\tilde{\alpha}_k$ in the k-th measurement, $P(\tilde{\alpha}_k|\tilde{\alpha}_{k-1}, ..., \tilde{\alpha}_1)$, can be determined from $P_{k-1}(\alpha)$, using Eqs. (10)-(13) and (47)-(49). This allows us to calculate the statistics of the $\hat{a}_{1,s}$ measurement results. The center position of the initial probability density of $\hat{a}_{1,s}$, α_0, can be estimated using a linear function of the measurement results, because

$$\langle\tilde{\alpha}_k\rangle = \int d\tilde{\alpha}_1\, P(\tilde{\alpha}_1) \int d\tilde{\alpha}_2\, P(\tilde{\alpha}_2|\tilde{\alpha}_1) ... \int d\tilde{\alpha}_k\, P(\tilde{\alpha}_k|\tilde{\alpha}_{k-1}, ..., \tilde{\alpha}_1)\, \tilde{\alpha}_k = \alpha_0 \quad , \tag{50}$$

However, the width of the initial probability density of $\hat{a}_{1,s}$, Δ_0^2, cannot be estimated using a quadratic function of the measurement results, because the information about Δ_0^2 is always "screened" by the α_0^2, which is unknown to us:

$$\begin{aligned} \langle\tilde{\alpha}_k^2\rangle &= \int d\tilde{\alpha}_1\, P(\tilde{\alpha}_1) \int d\tilde{\alpha}_2\, P(\tilde{\alpha}_2|\tilde{\alpha}_1) ... \int d\tilde{\alpha}_k\, P(\tilde{\alpha}_k|\tilde{\alpha}_{k-1}, ..., \tilde{\alpha}_1)\, \tilde{\alpha}_k^2 \\ &= \alpha_0^2 + \Delta_0^2 + \Delta_m^2 + (k-1)\Delta_b^2 \quad , \end{aligned} \tag{51}$$

$$\begin{aligned} \langle\tilde{\alpha}_k\tilde{\alpha}_l\rangle &= \int d\tilde{\alpha}_1\, P(\tilde{\alpha}_1) ... \int d\tilde{\alpha}_k\, P(\tilde{\alpha}_k|\tilde{\alpha}_{k-1}, ..., \tilde{\alpha}_1)\, \tilde{\alpha}_k ... \\ &\quad \times \int d\tilde{\alpha}_l\; P(\tilde{\alpha}_l|\tilde{\alpha}_{l-1}, ..., \tilde{\alpha}_k, ..., \tilde{\alpha}_1)\, \tilde{\alpha}_l \\ &= \alpha_0^2 + \Delta_0^2 + (k-1)\Delta_b^2 \quad , \quad \text{for all } k < l \ . \end{aligned} \tag{52}$$

The same treatment can be repeated using the measurement results of $\hat{a}_{2,s}$. In fact, comparing Eqs. (50)-(52) with Eqs. (23)-(25) we see that in this specific example, we recover the statistics of the general case.

The changes in the wavefunction of a single squeezed state due to a series of alternating QND measurements of the two quadrature amplitudes seem to be different from the changes due to a series of photon-number QND measurements. Yet, in both cases, the statistics of the measurement results do not contain any information about the uncertainties associated with the measured observables. Therefore, in both cases, these changes inhibit the measurement of the wavefunction. This is because the changes in the wavefunction due to any process of repeated quantum measurements are always governed by the projection postulate.

CONCLUSIONS

We have shown that the wavefunction of a single system cannot be inferred from the results of repeated quantum measurements. Mathematically, this is because each measurement result depends on the results of all the previous measurements. Physically, the measurement process modifies the wavefunction of the measured system in accordance with the measurement result, i.e., the information which is extracted from the system. This modification is a direct consequence of the projection postulate. Therefore, we conclude that the projection postulate inhibits the measurement of the quantum wavefunction of a single system, and limits this wavefunction to have a statistical (or epistemological) meaning only.

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18. Throughout this paper $N[x, x_0, \sigma^2]$ is defined to be a normal distribution (or a Gaussian) of the variable x, which is centered at x_0 with the variance σ^2,

$$N[x, x_0, \sigma^2] = (2\pi\sigma^2)^{-1/2} \exp[-(x - x_0)^2/(2\sigma^2)] \ .$$

19. See, e.g., A. Papoulis, **Probability, Random Variables and Stochastic Processes**, (Mc-Graw Hill Inc., New York, NY, 1991).
20. Throughout this paper $\chi^2[x, \nu]$ is defined to be a chi-squared distribution of the variable x, which is centered at ν with the variance 2ν,

$$\chi^2[x, \nu] = 2^{\nu/2}\Gamma(\nu/2)^{-1} x^{(\nu-2)/2} \exp(-x/2) \ .$$

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