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RAPID COMMUNICATIONS

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Protective measurement of the wave function of a single squeezed harmonic-oscillator state

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A scheme for the ''protective measurement'' [Phys. Rev. A **47**, 4616 (1993)] of the wave function of a squeezed harmonic-oscillator state is described. This protective measurement is shown to be equivalent to a measurement of an ensemble of states. The protective measurement, therefore, allows for a definition of the quantum wave function on a single system. Yet, this equivalency also suggests that both measurement schemes account for the epistemological meaning of the wave function only. The protective measurement requires a full *a priori* knowledge of the measured state. The intermediate cases, in which only partial *a priori* information is given, are also discussed.

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I. INTRODUCTION

Recently, with the technological developments that allow manipulation of single quantum systems, there has been an interest in the quantum theory of a single system. Traditionally, quantum mechanics describes a single system by a corresponding wave function. The quantum wave function, though, is defined on an ensemble of quantum systems, in the sense that the wave function could be fully determined from the results of measurements performed on this ensemble [1,2]. Naturally, one would ask the question: Can we define the quantum wave function on a single system? Aharonov, Anandan, and Vaidman [3,4] showed recently, that the wave function of a single system could be determined from the results of a series of ''protective measurements'' performed on the system. The measurement of a quantum system, i.e., a signal system, is composed of three stages: preparation of a probe system, interaction of the probe with the signal, and measurement (which induces collapse) of the probe. Usually, the probe and the signal are entangled after their interaction, and the collapse of the probe changes the wave function of the signal. It was recently shown by us [5] that this entanglement inhibits the measurement of the wave function of the signal. Using the results of repeated measurements performed on the signal, one could estimate the expectation values (or averages) of the various measured observables with finite estimate errors. One cannot, however, estimate the uncertainties (or variances) of these observables. Mathematically, this is because each measurement result depends on the results of all previous measurements performed on the signal. Physically, this is due to the change of the wave function of the signal each time a measurement is performed, in accordance with the measurement result. The ''protective measurement'' suggested by Aharonov, Anandan, and Vaidman is a measurement in which the signal and the probe are left disentangled after their interaction, and the proceeding measurement of the probe does not affect the signal at all. The change in the wave function of the signal is deterministic, because it is independent of the measurement result. This change can be very small when, for example, the interaction of the probe and the signal is weak. The protective measurement requires *a priori* knowledge of the wave function of the signal. A series of protective measurements performed on a single quantum system will allow us to confirm this *a priori* knowledge.

In this Rapid Communication we describe a scheme for the protective measurement of a squeezed harmonic-oscillator state. The *a priori* knowledge of the noise distribution, i.e., the squeezing parameter, of the signal, is used in preparing a probe in the ground excitation, i.e., in a vacuum state, with the ''opposite'' squeezing, to avoid entanglement of the probe and the signal as they couple linearly. The deterministic change in the wave function of the signal, a reduction in the excitation of the signal, is ''corrected for'' by driving the signal with a classical field back to its *a priori* known initial excitation. We consider a series of ''measurements without entanglement'' performed on the signal, where, after each measurement, the signal is driven back to its original state. In this scheme, the limits of weak and strong measurements correspond to a series of protective measurements of a single state and a measurement of an ensemble of states, respectively. We conclude that both cases are equivalent methods for the measurement of the wave function (of the single system or the ensemble of systems), which provide the same information and use the same experimental arrangement. The protective measurement allows, therefore, a definition of the quantum wave function on a single system, which is equivalent to the traditional definition on an ensemble of systems. Yet, the equivalency suggests that a definition of the wave function based on either measurement method can account only for the epistemological nature of the wave function. We also investigate the case in which only the squeezing parameter of the signal is known *a priori*. In this case the entanglement of the signal and the probe is avoided, but the deterministic reduction in the excitation of the signal cannot be corrected for. The optimal experimental setup, with the optimal choice of coupling constants, is analyzed for a series of measurements without entanglement performed on a single state. Using these measurement results it is possible to estimate the expectation values of the measured observables, with the minimum possible estimate errors being the initial uncertainties of the observables. These uncertainties can be estimated as accurately as we want, to confirm the *a priori* known noise distribution of the signal. We conclude that no information could be obtained about the wave function of a single system beyond the information that is provided by a single strong measurement of this system and the information that is given *a priori* to the measurement process.

II. MEASUREMENT WITHOUT ENTANGLEMENT

The squeezed harmonic-oscillator state, $|\alpha,r\rangle_s$, is an eigenstate of the operator $e^r\hat{s}_1+ie^{-r}\hat{s}_2$, where $\hat{s}_1$ and $\hat{s}_2$ are the generalized position and momentum of the harmonic oscillator. This signal state is defined by its excitation $|\alpha|^2$ and its squeezing parameter r, which determines the noise distribution of the signal. Note that in general r is a complex number, and the squeezed state is not a minimum uncertainty state. In our measurement scheme, the signal is being coupled linearly to a squeezed vacuum probe $|0,q\rangle_p$, where q is the squeezing parameter of the probe (Fig. 1). This interaction is described by the Hamiltonian $\hat{H}=\hbar\kappa(\hat{s}^\dagger\hat{p}+\hat{s}\hat{p}^\dagger)$, where $\hat{s}$, $\hat{s}^\dagger$ and $\hat{p}$, $\hat{p}^\dagger$ are the annihilation and creation operators of the signal and the probe, respectively. The coupling constant κ and the interaction time t define the ''transmission coefficient'' of this interaction, $T=\cos^2(\kappa t)$. In the Heisenberg picture, the time evolution of the signal and the probe due to their interaction is described by the relations

$$\hat{s}_{out}=\sqrt{T}\hat{s}_{in}-i\sqrt{1-T}\hat{p}_{in}, \tag{1}$$

$$\hat{p}_{out}=\sqrt{T}\hat{p}_{in}-i\sqrt{1-T}\hat{s}_{in}, \tag{2}$$

where $\hat{s}_{in}$, $\hat{p}_{in}$ and $\hat{s}_{out}$, $\hat{p}_{out}$ are the annihilation operators of the signal and the probe, before and after the interaction,

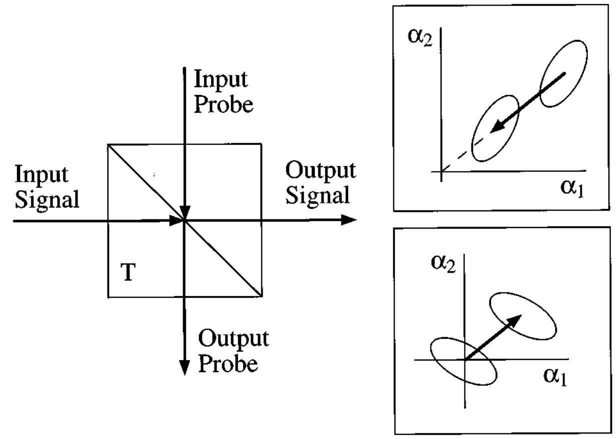

FIG. 1. Measurement without entanglement of a squeezed state of light: The signal and the probe, with opposite squeezing parameters, interact linearly in a beam splitter. The top and bottom insets show the changes in the signal and the probe, respectively. The excitation of the signal is reduced, while the excitation of the probe is increased.

respectively. A measurement of the output probe state $\hat{p}_{out}$, therefore, gives information about the input signal state $\hat{s}_{in}$.

To obtain information about the generalized position of the input signal, for example, $\hat{s}_{1,in}=(\hat{s}_{in}+\hat{s}_{in}^\dagger)/2$, one can measure the generalized momentum of the output probe, $\hat{p}_{2,out}=(\hat{p}_{out}-\hat{p}_{out}^\dagger)/2i=\sqrt{T}\hat{p}_{2,in}-\sqrt{1-T}\hat{s}_{1,in}$. The observed position,

$$\hat{s}_{1,obs}\equiv-\hat{p}_{2,out}/\sqrt{1-T}, \tag{3}$$

is centered at $\langle\hat{s}_{1,obs}\rangle=\langle\hat{s}_{1,in}\rangle$. The uncertainty of the observed position is the sum of the uncertainty of the position of the input signal and the measurement error,

$$\langle\Delta\hat{s}_{1,obs}^2\rangle=\langle\Delta\hat{s}_{1,in}^2\rangle+\frac{T}{1-T}\langle\Delta\hat{p}_{2,in}^2\rangle. \tag{4}$$

Similarly, a measurement of the position of the output probe $\hat{p}_{1,out}$ gives information about the momentum of the input signal $\hat{s}_{2,in}$. Note that, regardless of the specific observable that is being measured, when the coupling is highly ''transmissive,'' $T\approx1$, the measurement error is large and the measurement is weak. When the coupling is highly ''reflective,'' $T\approx0$, the measurement is strong.

The signal-probe interaction causes a deterministic change in the wave function of the signal, as can be seen from the above analysis. In general, though, the signal and the probe are entangled after the interaction, and a measurement of the probe would induce further change in the wave function of the signal, a stochastic change that depends on the measurement result. To find the special cases in which the signal and the probe are disentangled after their interaction one should examine their time evolution in the Schrödinger picture. Using normal ordering of the unitary time evolution operator, $\hat{U}(t)=\exp(-i\hat{H}t/\hbar)$, it can be shown [6] that when the input signal and probe are coherent states, $|\beta\rangle_s$ and $|\gamma\rangle_p$, the output signal and probe are disentangled coherent states (of different excitations),

$$\hat{U}(t)|\beta\rangle_s|\gamma\rangle_p=|\sqrt{T}\beta-i\sqrt{1-T}\gamma\rangle_s|\sqrt{T}\gamma-i\sqrt{1-T}\beta\rangle_p.$$

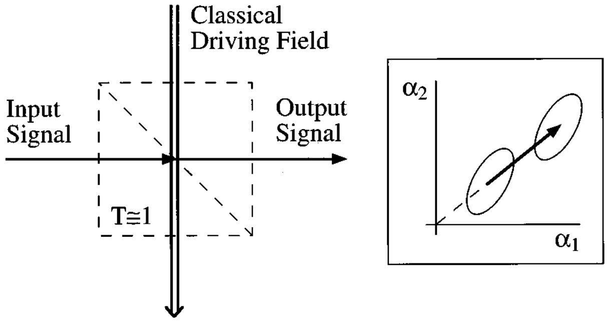

FIG. 2. Driving a squeezed state of light to its initial excitation: The signal interacts with a highly excited coherent state in a highly transmissive beam splitter. The inset shows the change in the signal.

This result can be used to determine the time evolution of the squeezed signal and the squeezed vacuum probe, writing the squeezed states in the coherent states representation,

$$\hat{U}(t)|\alpha,r\rangle_s|0,q\rangle_p=\int (d^2\beta/\pi)_s\langle\beta|\alpha,r\rangle_s \times\int (d^2\gamma/\pi)_p\langle\gamma|0,q\rangle_p\hat{U}(t)|\beta\rangle_s|\gamma\rangle_p .$$

This leads (after some math) to the conclusion that the output signal and probe are disentangled when their squeezing parameters r and q satisfy the relation $q=-r+i\phi$, where ϕ is an arbitrary phase. The squeezing of the probe is, therefore, required to be ''opposite'' to the squeezing of the signal,

$$\langle\Delta\hat{p}^2_{1,in}\rangle=\exp[-2\ \mathrm{Re}(r)]/4=\langle\Delta\hat{s}^2_{2,in}\rangle, \tag{5}$$

$$\langle\Delta\hat{p}^2_{2,in}\rangle=\exp[2\ \mathrm{Re}(r)]/4=\langle\Delta\hat{s}^2_{1,in}\rangle. \tag{6}$$

In this case, the disentangled output signal and probe are of different excitations but the same noise distributions as the input signal and probe, respectively,

$$\hat{U}(t)|\alpha,r\rangle_s|0,-r+i\phi\rangle_p =|\sqrt{T}\alpha,r\rangle_s|-i\sqrt{1-T}\alpha,-r+i\phi\rangle_p . \tag{7}$$

III. DRIVING THE SIGNAL BACK TO ITS INITIAL EXCITATION

After a measurement without entanglement takes place, the excitation of the signal is reduced, from $|\alpha|^2$ to $T|\alpha|^2$. The signal could be driven back to its initial excitation, if this excitation were known, using a classical field (Fig. 2). This interaction is described by the Hamiltonian $\hat{H}=i\hbar(f\hat{s}^\dagger-f^*\hat{s})$, where f is the classical field. The unitary time evolution operator that corresponds to this Hamiltonian is a displacement operator [7], $\hat{U}(t)=\exp[(f\hat{s}^\dagger-f^*\hat{s})t]=\hat{D}(ft)$. When acting on a coherent signal, the excitation of the signal is increased by ft,

$$\hat{U}(t)|\beta\rangle_s=\exp[(\beta^*f-\beta f^*)t/2]|\beta+ft\rangle_s ,$$

while its noise distribution is left unchanged. The same is true when the signal is a squeezed state. Expressing the squeezed signal in terms of coherent states,

$$\hat{U}(t)|\sqrt{T}\alpha,r\rangle_s=\int (d^2\beta/\pi)_s\langle\beta|\sqrt{T}\alpha,r\rangle_s\hat{U}(t)|\beta\rangle_s ,$$

one obtains (after some math) that the time evolution of the squeezed signal, when driven by a classical field, is

$$\hat{U}(t)|\sqrt{T}\alpha,r\rangle_s=\exp[\sqrt{T}(\alpha^*f-\alpha f^*)t/2]|\sqrt{T}\alpha+ft,r\rangle_s . \tag{8}$$

For the signal to be driven back to its original state (up to a phase factor), the field f and the interaction time t should be chosen to satisfy $\sqrt{T}\alpha+ft=\alpha$.

IV. THE PROTECTIVE MEASUREMENT OF A SINGLE STATE EQUIVALENT TO A MEASUREMENT OF AN ENSEMBLE

Given the wave function of the signal, it is possible to perform a series of measurements on the signal, such that each time a measurement is taken the signal is in its original known state. First, one performs a measurement without entanglement, where the knowledge about the noise distribution of the signal is used. The result of this measurement gives some information about the (already known) wave function of the signal. Then, the deterministic change in the signal is corrected for, using the *a priori* knowledge of the initial excitation of the signal, after which the wave function of the signal is (up to a phase factor) exactly the same as it was initially. A series of such measurements can give full information about this wave function and confirm our *a priori* knowledge. The most feasible realization of this measurement scheme is a measurement of the wave function of a squeezed state of light. For the measurement without entanglement, the squeezed signal could be coupled to a squeezed vacuum probe in a beam splitter, with the transmission coefficient T (Fig. 1). The effect of an interaction with a classical field could be achieved by using a highly excited coherent state (with a large signal-to-noise ratio), and causing it to interact with the signal in a highly transmissive beam splitter, $T\cong1$, so that the noise of the driving coherent state does not affect the squeezed signal (Fig. 2).

Now, consider the strength of these measurements. When the measurement without entanglement is strong, $T\approx0$, the deterministic change it causes to the signal is significant: The excitation of the signal is reduced from $|\alpha|^2$ to $T|\alpha|^2\approx0$. In this case, the process of driving the signal back to its original excitation is equivalent to preparing a new signal state. Therefore, in the limit of strong measurement, the series of measurements described above is, in fact, the case of preparing and measuring an ensemble of identical generalized harmonic-oscillator states. When the measurement is weak, $T\approx1$, the change in the excitation of the signal is almost negligible, $T|\alpha|^2\approx|\alpha|^2$. This limit is the case of repeated protective measurements performed on a single state. Since both cases, that of a series of protective measurements performed on a single state and that of a measurement of an ensemble of states, are limits of the same physical process, and, indeed, the same experimental arrangement, we conclude that these two cases are equivalent. And, of course, in these two cases the quantum wave function is measured.

This equivalency in the measurement of the wave function, establishes an equivalency between the two definitions of the wave function, the traditional definition that is based on an ensemble of systems and the new definition suggested by Aharonov, Anandan, and Vaidman that is based on a single system. The protective measurement requires a full *a priori* knowledge of the measured wave function. The measurement of an ensemble requires the knowledge of the experimental parameters that produce a quantum state. Due to the required *a priori* knowledge, both valid definitions of the wave function fail to account for the physical reality of the wave function of a single quantum system.

V. REPEATED MEASUREMENTS WITHOUT ENTANGLEMENT

In a measurement of the wave function of a single quantum system, to what extent is the *a priori* knowledge needed? To answer this question, consider the case of repeated measurements without entanglement of the generalized position $\hat{s}_1$ of a single generalized harmonic-oscillator state. Assume that, in order to perform these measurements, the noise distribution of the signal, i.e., $\langle \Delta \hat{s}_1^2 \rangle$ and $\langle \Delta \hat{s}_2^2 \rangle$, is known, but no additional information about the excitation of the signal, i.e., $\langle \hat{s}_1 \rangle$ and $\langle \hat{s}_2 \rangle$, is given. First consider the results of two consecutive measurements, $\tilde{s}_{1,1}$ and $\tilde{s}_{1,2}$. From Eq. (3), each result is, on average, $\langle \tilde{s}_{1,1} \rangle = \langle \tilde{s}_{1,2} \rangle = \langle \hat{s}_1 \rangle$. From Eqs. (4), (6), and (7), the errors associated with these results are $\langle \Delta \tilde{s}^{\,2}_{1,1} \rangle = \langle \Delta \hat{s}_1^2 \rangle/(1-T_1)$ and $\langle \Delta \tilde{s}^{\,2}_{1,2} \rangle = \langle \Delta \hat{s}_1^2 \rangle / T_1/(1-T_2)$. Define the estimate of $\langle \hat{s}_1 \rangle$ to be $\overline{s_1} = (\tilde{s}_{1,1} + \tilde{s}_{1,2})/2$, where $\langle \overline{s_1} \rangle = \langle \hat{s}_1 \rangle$. The measurement results, $\tilde{s}_{1,1}$ and $\tilde{s}_{1,2}$, are independent of each other, since the signal and probe are disentangled after their interaction, and the estimate error is $\langle \Delta \overline{s}^{\,2}_1 \rangle = (\langle \Delta \tilde{s}^{\,2}_{1,1} \rangle + \langle \Delta \tilde{s}^{\,2}_{1,2} \rangle)/4$. This error is minimized when the transmission coefficients are chosen such that $T_1 = 1/2$ and $T_2 = 0$. In this case, the estimate error equals the initial uncertainty in the generalized position of the signal, $\langle \Delta \overline{s}^{\,2}_1 \rangle_{min} = \langle \Delta \hat{s}_1^2 \rangle$. To estimate this initial uncertainty using the measurement results, define $\sigma^2 = (\tilde{s}_{1,1} - \tilde{s}_{1,2})^2/4$, where, for the above choice of T_1 and T_2, $\langle \sigma^2 \rangle = \langle \Delta \hat{s}_1^2 \rangle$. The error in the uncertainty estimate is $\langle \Delta(\sigma^2)^2 \rangle = 2\,\langle \Delta \hat{s}_1^2 \rangle$, where we used the fact that the probablity densities of the generalized position and momentum of a generalized harmonic oscillator state are Gaussians.

This analysis can be generalized for n measurement results by way of mathematical induction. In this case, the estimate of the generalized position of the signal is defined as

$$\overline{s_1} = \frac{1}{n} \sum_{k=1}^{n} \tilde{s}_{1,k}. \tag{9}$$

The associated estimate error is minimized when the transmission coefficients are

$$T_1 = \frac{n-1}{n},\ \ T_2 = \frac{n-2}{n-1}, \ldots,\ \ T_k = \frac{n-k}{n-k+1}, \ldots,\ \ T_n = 0. \tag{10}$$

The minimum possible estimate error always equals the initial uncertainty of the position of the signal, $\langle \Delta \overline{s}^{\,2}_1 \rangle_{min} = \langle \Delta \hat{s}_1^2 \rangle$, regardless of the number of measurements. However, the error in the estimate of the initial uncertainty,

$$\sigma^2 = \frac{1}{(n-1)n^2} \sum_{k=1}^{n} \sum_{l>k} (\tilde{s}_{1,k} - \tilde{s}_{1,l})^2, \tag{11}$$

is reduced as the number of measurement increases: $\langle \Delta(\sigma^2)^2 \rangle = 2\,\langle \Delta \hat{s}_1^2 \rangle^2/(n-1)$. Note that this is the same error as when $\langle \Delta \hat{s}_1^2 \rangle$ is estimated using n measurement results obtained from an ensemble of n identical squeezed harmonic-oscillator states. Repeated measurements without entanglement of a single state, therefore, give the same information on the (unknown) excitation of the signal as a single strong measurement does. The (known) noise distribution of the signal can be determined with increasing accuracy, as the number of measurements increases. We conclude that in order to obtain information about the wave function, which a single strong measurement cannot give, this information is required to be known *a priori* to the measurement process.

VI. CONCLUSIONS

We have described a scheme for the protective measurement of a squeezed harmonic-oscillator state. Using this scheme, we have shown that the protective measurement of a single system is equivalent to a measurement of an ensemble of systems. Therefore, the protective measurement allows for a definition of the quantum wave function on a single system. Yet, the protective measurement of a single system accounts only for the epistemological nature of the wave function of the single system, and does not add to it physical reality. Analyzing the case in which only partial *a priori* information about the wave function of the system is available, we have shown that without any *a priori* knowledge, the information that can be extracted from a single system is limited to the information obtained with a single strong measurement.

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[1] See, e.g., K. Vogel and H. Risken, Phys. Rev. A **40**, 2847 (1989).

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