Can We Measure the Wave Function of a Single Wave Packet of Light?

Brownian Motion and Continuous Wave Packet Collapse in Repeated Weak Quantum Nondemolition Measurements

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Two separate structures exist in quantum mechanics: the observables, which are represented by operators of the Hilbert space, and the physical systems, which are described by state vectors or wave functions. The wave function is said to have an epistemological meaning because it contains all the relevant information about the physical system under consideration. The result of a precise measurement on a single quantum system is always one of the eigenvalues of the measured observable. After the measurement, the wave function of the measured system collapses to the corresponding eigenstate, according to the projection postulate. There is no one-to-one correspondence between the result of a single measurement and the state of the system before the measurement. In order to measure the initial wave function of the system, one needs to prepare an ensemble of systems with the same wave function and then measure them all. The wave function is obtained from the statistics of the results of measurements performed on this ensemble. Recently, Aharonov, Anandan, and Vaidman[1,2] suggested that the wave function of a single quantum system could be measured, thereby giving the wave function an ontological significance, that is, physical reality in its own right, in addition to its usual epistemological role. They suggested employing a series of "protective measurements", where an a priori knowledge of the wave function enables one to measure this wave function and to protect it from changing at the same time. However, with this a priori knowledge, one could reproduce the wave function after each measurement for an arbitrarily large number of times and one could then measure the wave function in the conventional manner.

In this report, we investigate the possibility of measuring the wave function of a single quantum system with no a priori knowledge of the wave function in order to explore a real ontological meaning of the wave function. We study the case of repeated weak quantum nondemolition (QND) measurements,[3,4] for which we can assume that the signal and the probe are in pure states before the measurement, without loss of generality. In this case, the signal is left in a pure state after the measurement. The unitary interaction between the probe and the signal does not allow transitions between any two eigenstates of the measured observable, where such transitions cause an appreciable change of the wave function. Also, the QND measurement can be chosen to be as weak as we want. It is possible, therefore, to measure the signal many times, using weak QND measurements, before the wave function of the signal is changed significantly. The measurement results are all generated under some influence of the initial wave function and one may expect the statistics of these results to give at least partial information about this wave function. In this work, we show that this intuitive picture fails and one cannot, in fact, extract any information about the initial wave function of the signal at all. By "information about the wave function", we mean information about both the average and the variance of the measured observable, that is, the center and the width of the wave packet, with finite probability errors. Information about the center position alone corresponds to a measurement of the observable, where information about the variance reveals the wave function.

In our model, a series of photon-number QND measurements is performed on a single wave packet of light.[5-8] A signal wave packet of light, $|\psi_0\rangle_s$, is correlated to a probe wave packet, $|\alpha_0\rangle_p$, in an optical Kerr medium. This process is described by the unitary operator $\hat{U} = \exp(i\mu\hat{n}_s\hat{n}_p)$, where $\hat{n}_s$ and $\hat{n}_p$ are the signal and probe photon-number operators, respectively, and μ is the coupling strength.[9] The photon-number of the signal, $\hat{n}_s$, shifts the phase of the probe, $\Delta\hat{\phi}_p \cong \mu\hat{n}_s$. Then, the second-quadrature amplitude of the probe, $\hat{a}_2$, is measured precisely by a homodyne detection. The inferred signal photon-number, $\tilde{n}$, is obtained from α_2^1, that is, the result of the probe quadrature measurement, $\tilde{n}_1 \cong \alpha_2^1/(|\alpha_0|\mu)$, where $|\alpha_0|$ is the initial excitation of the probe. A back-action noise is imposed on the phase of the signal by the probe photon-number, but this noise does not influence the photon-number distribution of the signal. The probability-amplitude operator, $\hat{Y}_1 = {}_p\langle\alpha_2^1|\hat{U}|\alpha_0\rangle_p$, completely describes the three stages of this QND measurement:[10] the preparation of the probe state, $|\alpha_0\rangle_p$; the interaction of this state with the signal, $\hat{U}$; and the results of the measurement, α_2^1, which corresponds to the state of the probe after the measurement, $|\alpha_2^1\rangle_p$. The probability of obtaining α_2^1 as the readout of the homodyne detection is $P(\alpha_2^1) = \mathrm{Tr}_s[\hat{Y}_1^+\hat{Y}_1\hat{\rho}_0]$, where $\hat{\rho}_0 = |\psi_0\rangle_s\,{}_s\langle\psi_0|$ is the density operator of the signal before the measurement. After a homodyne detection, which results in α_2^1, the signal density operator becomes $\rho_1 = P(\alpha_2^1)^{-1}\hat{Y}_1\hat{\rho}_0\hat{Y}_1^+$ and the corresponding photon-number distribution is $P_1(n) = {}_s\langle n|\hat{\rho}_1|n\rangle_s$.

The same measurement procedure is repeated k times. Each time, the measurement is performed on the output signal of the previous measurement, using a new probe state. We get a series of second-quadrature amplitude readouts, $(\alpha_2^1, \alpha_2^2, \ldots, \alpha_2^k)$, which correspond to a series of inferred photon-number values, $(\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_k)$. It is the statistics of $(\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_k)$, in the limit of weak measurements, that are expected to give the initial photon-number distribution of the signal, $P_0(n) = {}_s\langle n|\hat{\rho}_0|n\rangle_s$. The probability of obtaining a specific series of inferred photon-number values is

$$P(\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_k)\prod_{i=1}^{k} d\tilde{n}_i = P(\alpha_2^1, \alpha_2^2, \ldots, \alpha_2^k)\prod_{i=1}^{k} d\alpha_2^i, \tag{1}$$

where $P(\alpha_2^1, \alpha_2^2, \ldots, \alpha_2^k) = \mathrm{Tr}_s[\hat{Z}_k^+\hat{Z}_k\hat{\rho}_0]$ and where $\hat{Z}_k = \hat{Y}_k\ldots\hat{Y}_2\hat{Y}_1$ is the total probability-amplitude operator that describes the whole process of k repeated QND measurements.[10] The photon-number distribution of the signal wave function after the k-th measurement, $P_k(n) = {}_s\langle n|\hat{\rho}_k|n\rangle_s$, is calculated from the corresponding signal density operator,

$$\hat{\rho}_k = P(\alpha_2^1, \alpha_2^2, \ldots, \alpha_2^k)^{-1}\hat{Z}_k\hat{\rho}_0\hat{Z}_k^+. \tag{2}$$

Note that $\hat{Z}_k$ is symmetric in $(\alpha_2^1, \alpha_2^2, \ldots, \alpha_2^k)$; that is, it is independent of the order in which the results are obtained. This is because the different $\hat{Y}_i$ operators commute with each other. Therefore, the probability of obtaining these results, $P(\alpha_2^1, \alpha_2^2, \ldots, \alpha_2^k)$, and the final photon-number distribution after these results are measured, $P_k(n)$, are both symmetric in $(\alpha_2^1, \alpha_2^2, \ldots, \alpha_2^k)$. The probability of obtaining α_2^2 in the second measurement depends on the result of the first measurement, α_2^1. Yet, the process of measuring α_2^1 first and α_2^2 second has exactly the same probability as the process in which α_2^2 is measured first and α_2^1 is measured second. Also, there is no inherent difference between the changes caused to the wave function by the different consecutive measurements. Because the wave function of the system is slightly different at each measurement, the above observation, namely, that $\hat{Z}_k$ is independent of the order of the measurement results, suggests that no information about the width of the wave function is contained in the statistics of the readouts, $P(\alpha_2^1, \alpha_2^2, \ldots, \alpha_2^k)$.

To confirm this, let us assume that the initial photon-number distribution is a Gaussian, that is, a normal distribution,[11] $P_0(n) = N[n,n_0,\delta_0^{-1}]$. Physically, the photon-number distribution is a discrete distribution, where $n \geq 0$. If the signal is initially in a squeezed state with a large excitation, that is, $n_0 \gg 1$, this Gaussian approximation is valid. The initial distribution of the second-quadrature amplitude of the probe, with the probe being in a squeezed state with a zero phase, is also a Gaussian, centered at zero with the variance $\langle \Delta\hat{a}_2^2\rangle = e^{-2r}/4$, where r is the squeezing parameter. Our model describes a measurement process in which both the signal and the probe have normal distributions. Many other physical schemes are described in the same way—for example, the QND measurement of one of the quadrature amplitudes of a wave packet of light, using a nondegenerate parametric amplification.[12,13] Using $\alpha_2^i \cong |\alpha_0|\mu\hat{n}_i$ in equations 1 and 2, the probability distribution for inferring a series of photon-number values is

$$P(\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_k) = \int_{-\infty}^{\infty} dn\, N[n, n_0, \delta_0^{-1}] \prod_{i=1}^{k} N[\tilde{n}_i, n, \delta_m^{-1}]. \tag{3}$$

The final photon-number distribution of the signal is

$$P_k(n) = P(\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_k)^{-1} N[n, n_0, \delta_0^{-1}] \prod_{i=1}^{k} N[\tilde{n}_i, n, \delta_m^{-1}]. \tag{4}$$

Here, $\delta_m = |\alpha_0|^2\mu^2/\langle\Delta\hat{a}_2^2\rangle$ is the strength of each consecutive measurement. The measurements are weak when the error associated with each measurement is much larger than the initial width of the photon-number distribution of the signal, that is, when $\delta_m \ll \delta_0$.

Consider the case of one measurement performed on a single wave packet of light. The probability distribution of measuring the inferred photon-number, $\hat{n}_1$, is

$$P(\tilde{n}_1) = N[\tilde{n}_1, n_0, \delta_0^{-1} + \delta_m^{-1}]. \tag{5}$$

The signal wave function is changed to

$$P_1(n) = N[n, n_0^1, (\delta_0 + \delta_m)^{-1}], \tag{6}$$

$$n_0^1 = (\delta_0 n_0 + \delta_m \tilde{n}_1)(\delta_0 + \delta_m)^{-1}. \tag{7}$$

The center of the wave function, n_0^1, is shifted toward the measurement result, $\tilde{n}_1$, from its original value, n_0, whereas the width of the wave function narrows from δ_0^{-1} to $(\delta_0 + \delta_m)^{-1}$. If the measurement is weak, that is, $\delta_m \ll \delta_0$, both the shift and the narrowing are very small.

Before investigating the case of repeated QND measurements performed on a single wave packet, we analyze the case of one measurement performed on each wave packet in an ensemble of k wave packets, all prepared in the same initial state. In this case, each measurement is independent of the others. The probability of obtaining the inferred photon-number values, $(\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_k)$, is obviously independent of their order, $P(\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_k) = \Pi_{i=1}^{k} P(\tilde{n}_i)$. It is well known that the statistics of the results of the measurements in this case are analyzed by both the inferred average $\bar{n} = \Sigma_{i=1}^{k} \tilde{n}_i/k$ and the inferred variance $\overline{\Delta n^2} = \Sigma_{i=1}^{k} (\tilde{n}_i - \bar{n})^2/(k - 1)$, in which all measurement results have the same weight. In terms of $\bar{n}$ and $\overline{\Delta n^2}$, the probability that the measurements performed on the ensemble would result in $(\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_k)$ is

$$\prod_{i=1}^{k} P(\tilde{n}_i)d\tilde{n}_i = [P(\bar{n})dn][P(S)dS]d\Omega_{k-1}, \tag{8}$$

where $S = (k - 1)(\delta_0^{-1} + \delta_m^{-1})^{-1}\overline{\Delta n^2}$ and $d\Omega_{k-1}$ is a normalized infinitesimal element of the solid angle in dimension $(k - 1)$, that is, $\int d\Omega_{k-1} = 1$. The probability distribution of the inferred average is

$$P(\bar{n}) = N[\bar{n}, n_0, k^{-1}(\delta_0^{-1} + \delta_m^{-1})]. \tag{9}$$

$P(\bar{n})$ is centered at the original center of the wave function, n_0. Therefore, the inferred average, $\bar{n}$, is a statistical measure of n_0. The variance of $P(\bar{n})$ is inversely proportional to the number of measurements, k. The probability error associated with this measurement decreases as the number of measurement results increases. The probability distribution of S is a chi-square distribution,[14] $P(S) = \chi^2[S, (k - 1)]$. Therefore, the distribution of the inferred variance, $\overline{\Delta n^2}$, is centered at $\delta_0^{-1} + \delta_m^{-1}$, with the variance $2(k - 1)^{-1}(\delta_0^{-1} + \delta_m^{-1})^2$. As k increases, the probability error for $\overline{\Delta n^2}$ to read $\delta_0^{-1} + \delta_m^{-1}$ decreases. By measuring an ensemble of wave packets, all with the same initial wave function, we can conclude that both the center of the wave function and its width can be inferred statistically. This corresponds to a measurement of the wave function.

Next, let us consider the changes in the measured wave function in the process of k repeated measurements performed on a single wave packet. From equation 4, we obtain that the final photon-number distribution after k repeated measurements, which results in $(\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_k)$, is

$$P_k(n) = N[n, n_0^k, (\delta_0 + k\delta_m)^{-1}], \tag{10}$$

$$n_0^k = \left(\delta_0 n_0 + \delta_m \sum_{i=1}^{k} \tilde{n}_i\right)(\delta_0 + k\delta_m)^{-1}. \tag{11}$$

As was noted before, $P_k(n)$ is symmetric in $(\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_k)$. Also, by comparing equations 10 and 11 with equations 6 and 7, the total change in the wave function due to k repeated measurements of strength δ_m is exactly the same as the change due to one measurement of a strength $k\delta_m$, which results in $\tilde{n} = \bar{n}$. After each measurement, the width of the wave packet decreases (continuous wave packet collapse). The center of the wave packet takes a step in a random walk (quantum Brownian motion), which depends on the random result of the measurement, $\tilde{n}_i$. The probability distribution that statistically describes the diffusion of the center position of the wave function after k measurements, n_0^k, is

$$P_k(n_0^k) = N[n_0^k, n_0, k\delta_m\delta_0^{-1}(\delta_0 + k\delta_m)^{-1}]. \tag{12}$$

The average center position is always at the initial center position, n_0. However, the probability of finding the center further away from n_0 increases as the number of measurements increases. As long as the total strength of the measurements is small, that is, $k\delta_m \ll \delta_0$, the variance of n_0^k increases linearly with the number of measurements, $k\delta_m\delta_0^{-1}(\delta_0 + k\delta_m)^{-1} \cong Dk$. In this regime, the movement of the center position is a quantum Brownian motion with a constant diffusion coefficient, $D = \delta_m\delta_0^{-2}$. Here, the time scale is replaced by the discrete scale of the number of measurements. As the wave function narrows, the average step size of this quantum Brownian motion decreases. The statistical variance of the center position saturates and then equals the original variance of the wave function, δ_0^{-1}. At the same time, the wave packet is reduced to a photon-number eigenstate. The measured wave packet, therefore, undergoes a quantum Brownian motion, which is saturated due to the continuous collapse of the wave packet.

Analyzing the statistics of the results of k repeated measurements on a single wave packet, we use the same definitions for the inferred average and variance as for the case of k measurements performed on an ensemble. Both definitions are symmetric in the results of the measurements, $(\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_k)$. In the case of k repeated measurements on a single wave packet, both the final wave function and the probability to obtain a specific series of results are independent of the order in which these results are obtained. Therefore, it is natural to use the same $\bar{n}$ and $\overline{\Delta n^2}$ as before. From equation 3, the probability of obtaining the series $(\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_k)$ as a result of k repeated measurements is

$$P(\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_k) \prod_{i=1}^{k} d\tilde{n}_i = [P(\bar{n})d\bar{n}][P(S)dS]d\Omega_{k-1}, \tag{13}$$

where $S = (k - 1)\delta_m\overline{\Delta n^2}$, and is independent of δ_0. Again, the probability distribution of the inferred average is centered at the original average, n_0, that is,

$$P(\bar{n}) = N[\bar{n}, n_0, \delta_0^{-1} + (k\delta_m)^{-1}]. \quad (14)$$

The variance of the inferred average decreases with an increased number of measurements and, as $k \to \infty$, this variance reaches its minimum value, which equals the original variance of the wave packet, δ_0^{-1}. Therefore, the inferred average has the same probability error in the cases of both an infinite number of repeated weak measurements and one precise measurement. In fact, comparing equation 14 with equation 5, we see that $\bar{n}$ is inferred with equal probabilities by k consecutive measurements of strength δ_m and by one measurement of a strength $k\delta_m$. The probability distribution of S is again a chi-square distribution, $P(S) = \chi^2[S, (k-1)]$. However, $P(S)$ is now independent of δ_0; therefore, the inferred variance, $\overline{\Delta n^2}$, is not a measure of the original variance, δ_0^{-1}. Indeed, $\overline{\Delta n^2}$ is centered at δ_m^{-1}, with the variance $2(k-1)^{-1}\delta_m^{-2}$. The statistics of the results of repeated weak QND measurements performed on a single wave packet contain no information about the initial width of the wave packet. In contradiction with our expectations, these statistics do not infer the wave function of the single wave packet.

The mathematical origin of this result is the symmetry of $P(\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_k)$, which appeared already in equation 1. Each time that the wave packet is measured, it is slightly changed. The results of the consecutive measurements are essentially collected from an ensemble of wave packets with different widths. Because all these results have the same weight in $P(\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_k)$, their statistics are independent of the width of the initial wave function. There is no natural way to assign different weights to the different results in the definition of $\overline{\Delta n^2}$ because the changes in the wave function are symmetric in $(\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_k)$ and we cannot overcome the symmetry of $P(\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_k)$.

Physically, it is the exact coordination between the quantum Brownian motion and the continuous collapse of the wave packet that prevents us from distinguishing between two wave packets of large and small widths, both centered at n_0. Probably, the first measurement result obtained from the wide wave packet is further away from n_0 than the result obtained from the narrow wave packet. However, the shift toward the measurement result and the collapse due to the first measurement are more dramatic in the case of the wide wave packet. Therefore, the probability of obtaining the second result in a certain distance from the first result can be the same for both wave packets, regardless of their initial widths.

The above result is consistent with the fundamental theorem of quantum communications,[15] namely, Holevo's theorem. The maximum channel capacity is realized by a photon-number state channel, in which the photon-number state signal is detected by an ideal photon counter.[16,17] The finite capacity of this noiseless channel is due to the discrete spectra of the photon-number; that is, the number of distinguishable states is finite because the photon-number observable has only positive integer eigenvalues. If one could measure the variance as well as the average of a given wave packet, the number of distinguishable states would increase by replacing the photon-number state with other states with the same average and varied variances. The possibility of exceeding the maximum channel capacity is excluded because the measurement of the average is subject to an error, which is determined by the initial variance of the wave packet, and the variance measurement is inhibited.

In conclusion, we have shown that the wave function of a single quantum system cannot be measured by a series of weak QND measurements without an a priori knowledge of the wave function. This is because the statistics of the results of the measurements contain no information about the initial width of the measured wave function. Mathematically, this result originates in the symmetric structure of the probability-amplitude operator. During the measurement process, the wave function undergoes a quantum Brownian motion and continuous collapse. This physical mechanism is responsible for the exact cancellation of the information about the wave function from the statistics of the measurement results.

REFERENCES AND NOTES

1. AHARONOV, Y. & L. VAIDMAN. 1993. Phys. Lett. A **178:** 38.
2. AHARONOV, Y., J. ANANDAN & L. VAIDMAN. 1993. Phys. Rev. A **47:** 4616.
3. BRAGINSKY, V. B., Y. I. VORONSTOV & K. S. THORNE. 1980. Science **209:** 547.
4. CAVES, C. M., K. S. THORNE, W. P. DREVER, V. D. SANDBERG & M. ZIMMERMANN. 1980. Rev. Mod. Phys. **52:** 341.
5. BRAGINSKY, V. B. & S. P. VYATCHANIN. 1981. Sov. Phys. Dokl. **26:** 686.
6. MILBURN, G. J. & D. F. WALLS. 1983. Phys. Rev. A **28:** 2065.
7. IMOTO, N., H. A. HAUS & Y. YAMAMOTO. 1985. Phys. Rev. A **32:** 2287.
8. IMAMOǦLU, A. 1993. Phys. Rev. A **47:** R4577.
9. KITAGAWA, K., N. IMOTO & Y. YAMAMOTO. 1987. Phys. Rev. A **35:** 5270.
10. BRAGINSKY, V. B. & Y. F. KHALILI. 1992. Quantum Measurement. Cambridge University Press. London/New York.
11. In this report, $N[x, x_0, \sigma^2]$ is defined to be a normal distribution of the variable x, which is centered at x_0 with the variance of σ^2: $N[x, x_0, \sigma^2] = (2\pi\sigma^2)^{-1/2} \exp[-(x - x_0)^2/2\sigma^2]$.
12. SHELBY, R. M. & M. D. LEVENSON. 1987. Opt. Commun. **64:** 553.
13. LAPORTA, A., R. E. SLUSHER & B. YURKE. 1989. Phys. Rev. Lett. **62:** 28.
14. In this report, $\chi^2[x, \nu]$ is defined to be a chi-square distribution of the variable x, which is centered at ν with the variance 2ν: $\chi^2[x, \nu] = 2^{\nu/2}\Gamma(\nu/2)^{-1}x^{(\nu-2)/2}\exp(-x/2)$.
15. HOLEVO (KHOLEVO), A. S. 1982. Probabilistic and Statistical Aspects of Quantum Theory. North-Holland. Amsterdam.
16. YAMAMOTO, Y. & H. A. HAUS. 1986. Rev. Mod. Phys. **58:** 1001.
17. CAVES, C. M. & P. D. DRUMMOND. 1994. Rev. Mod. Phys. **66:** 481.