Hierarchical Geodesic Models of Longitudinal Shape

Tom Fletcher, Prasanna Muralidharan, Nikhil Singh

School of Computing
Scientific Computing and Imaging Institute
University of Utah

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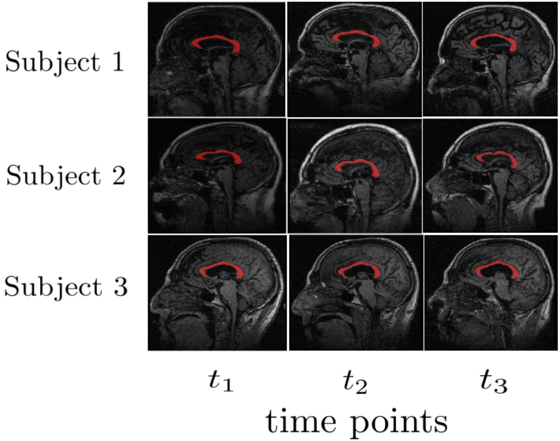
Longitudinal Shape Analysis

Goal: Understand how **individuals** change over time.



OASIS data:

72 healthy subjects

64 dementia subjects

2-5 images ∼1 year apart

http://wwww.oasis-brains.org

Linear Mixed-Effects Models

$$\begin{aligned} \text{Subject-level:} \quad & y_i = X_i\beta + Z_i b_i + \epsilon \\ \text{Group-level:} \quad & b_i \sim N(0, \Lambda) \end{aligned}$$

Data matrices: X_i, Z_i, typically with Z_i a subset of X_i

Fixed Effects (β): coefficients shared by all individuals
Random Effects (b_i): perturbation of ith individual

Laird and Ware, *Biometrics*, 1982

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Estimation by EM algorithm (b_i are latent variables)

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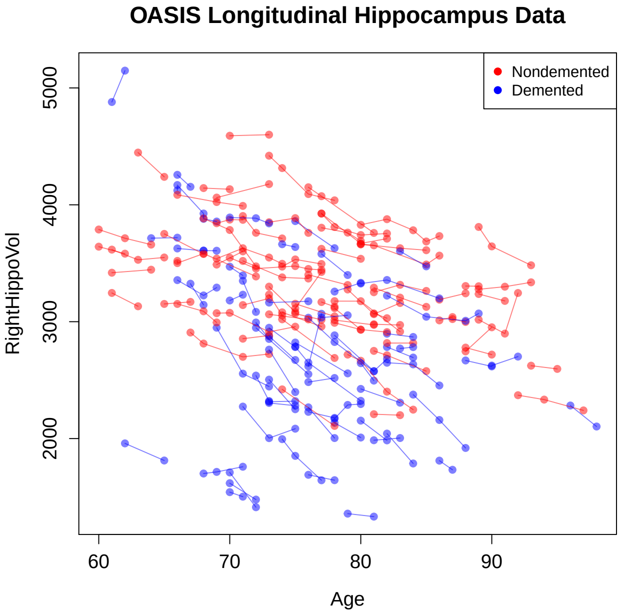
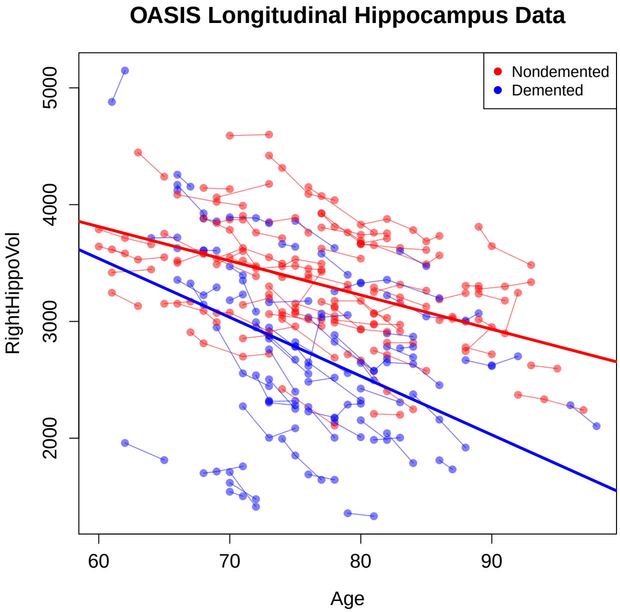
Fitting Linear Mixed-Effects Models in R

Scalar Data Example:

- Dependent variable: Right hippocampal volume
- Fixed effects: intercept, age slope, group effect
- Random effects: intercept

```
> lmeExample = lme(RightHippoVol ~ Age * Group,
+                  random = ~1 | ID, data = ldat)
```


OASIS Longitudinal Hippocampus Data
Nondemented
Demented
RightHippoVol
5000
4000
3000
2000
60
70
80
90
Age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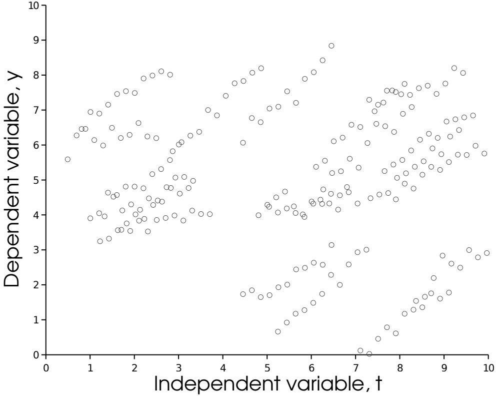

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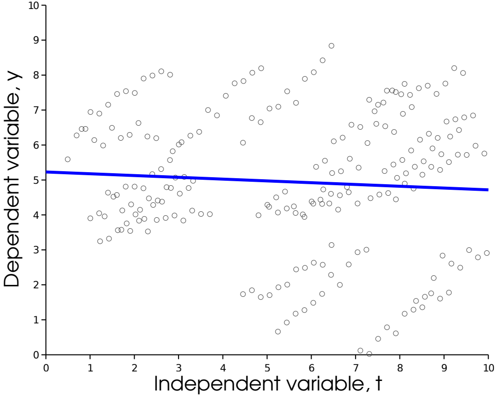

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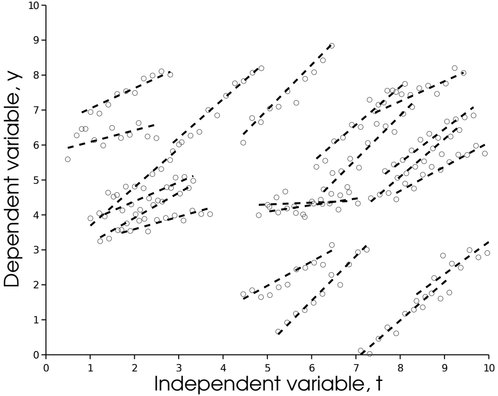

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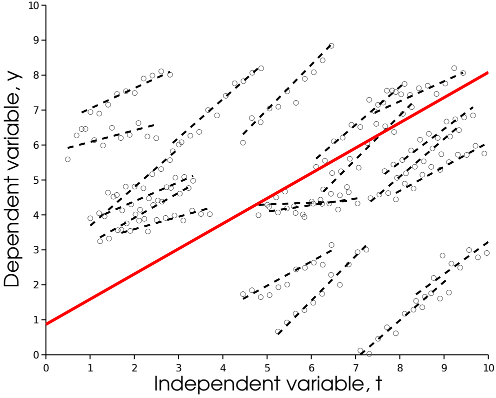

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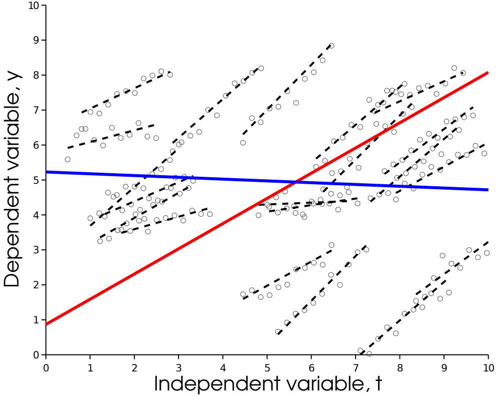
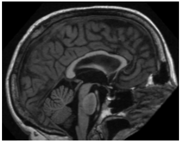


Shape Representations

Structure Boundaries (Kendall's Shape Space)



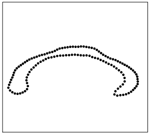
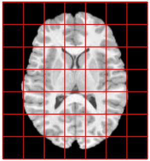
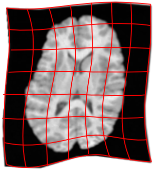


Image Deformations (Diffeomporphisms)





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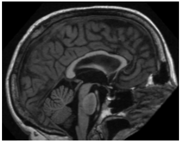

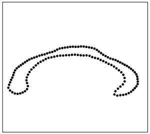

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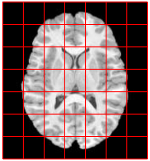

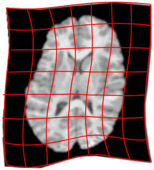

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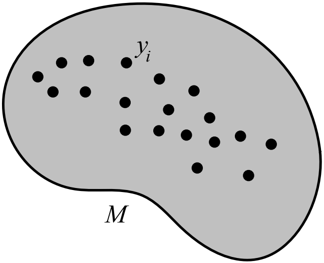
In both cases, data live on a **high-dimensional, nonlinear manifold**.

Regression on Manifolds



Given:

Manifold data: $y_i \in M$

Scalar data: $x_i \in \mathbb{R}$

Want:

Relationship $f : \mathbb{R} \to M$

"how x explains y"

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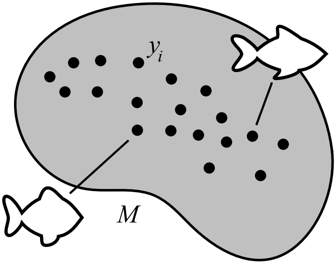

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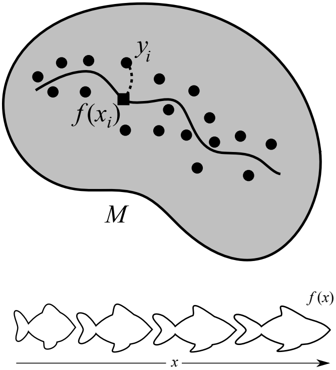

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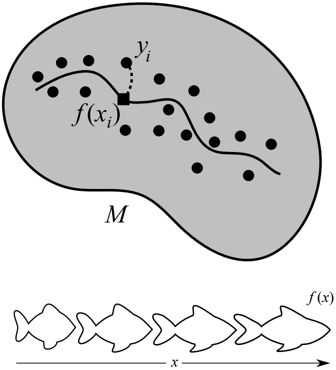

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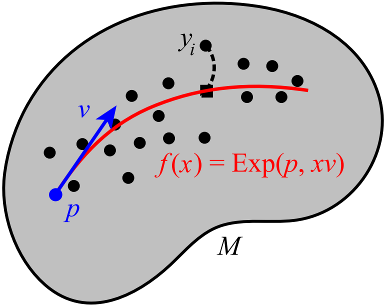
$$\hat{f} = \arg\min_{f} \sum_{i=1}^{N} d(f(x_i), y_i)^2$$

This is a **least squares** problem.

Geodesic Regression

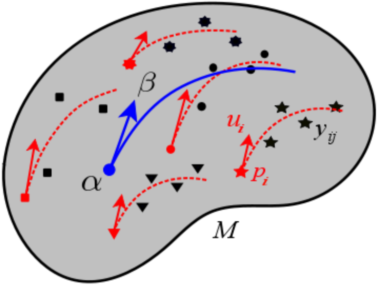


- Generalization of linear regression.
- Least-squares fitting of geodesic to the data (x_i, y_i).

$$(\hat{p}, \hat{v}) = \arg \min_{(p,v) \in TM} \sum_{i=1}^{N} d\left(\mathrm{Exp}(p, x_i\, v), y_i\right)^2$$

Fletcher, MFCA 2011, IJCV 2013; Niethammer et al., MICCAI 2011

Hierarchical Geodesic Models for Longitudinal Data



- **Group Level:** Average geodesic trend (α, β)
- **Individual Level:** Trajectory for ith subject (p_i, u_i)

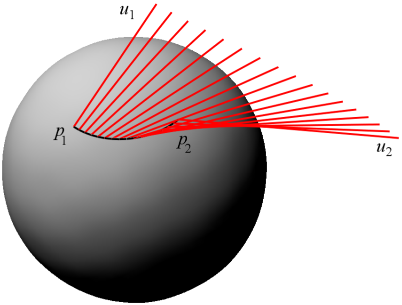
Muralidharan et al., CVPR 2012; Singh et al., IPMI 2013

Comparing Geodesics: Sasaki Metrics

What is the distance between two geodesic trends?

Define distance between initial conditions:

$$d_S((p_1, u_1), (p_2, u_2))$$



Sasaki geodesic on tangent bundle of the sphere.

Hierarchical Model Using The Sasaki Metric

$$
\begin{aligned}
y_{ij} &= \mathrm{Exp}(\mathrm{Exp}(p_i, x_{ij}u_i), \epsilon_{ij}) && \text{Individual Level} \\
(p_i, u_i) &= \mathrm{Exp}_S((\alpha, \beta), (v_i, w_i)) && \text{Group Level}
\end{aligned}
$$

where Exp is the exponential map on M and Exp_S is the exponential map on the tangent bundle TM, with respect to the Sasaki metric on TM.

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- This is feasible for **finite-dimensional** manifolds.

Hierarchical Model Using The Sasaki Metric

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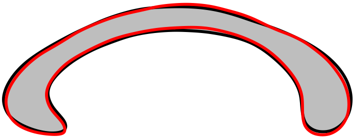
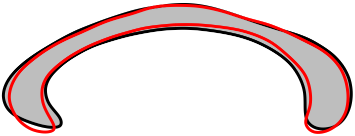
- This is feasible for **finite-dimensional** manifolds.
- **Diffeomorphisms**, not so much.

Results on Longitudinal Corpus Callosum



Non-Demented Trend

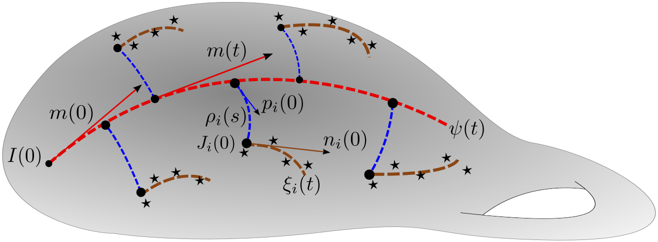


Demented Trend

Permutation Test:

Variable	T^2	p-value
Intercept α	0.734	0.248
Slope β	0.887	0.027

HGM for Diffeomorphisms



- Individual level: N geodesic regression problems
- Group level: One group geodesic, $I(0), m(0)$

Comparing Geodesics for Diffeomorphisms

Group level geodesic parameterization

- **Intercept:** Image: I
- **Slope:** Initial momenta field: $m = Lv$

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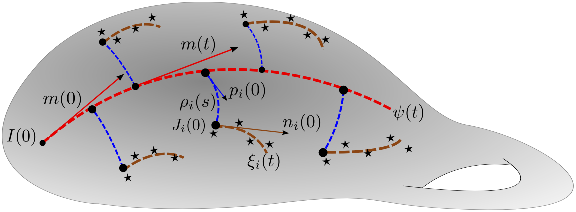
Transforming intercepts and slope

- Group action on image: $\phi \cdot I = I \circ \phi^{-1}$
- Group action on momenta:
  $$\phi \cdot m(0) = \underbrace{\mathrm{Ad}^*_{\phi^{-1}} m(0)}_{\text{Co-adjoint transport}}$$

Group Level Optimization Problem



$$
\begin{aligned}
\mathcal{E}(m(0), I(0), p_i(0)) = \; & \overbrace{\frac{1}{2}\|m(0)\|_K^2}^{\text{Distance metric for group}} \\
& + \overbrace{\frac{1}{2\sigma_I^2}\sum_{i=1}^{N}(\|p_i(0)\|_K^2 + \|\rho_i \cdot \psi(t_i) \cdot I(0) - J_i\|_{L^2}^2)}^{\text{Intercept match}} \\
& + \overbrace{\frac{1}{2\sigma_S^2}\sum_{i=1}^{N}\|\rho_i \cdot \psi(t_i) \cdot m(0) - n_i\|_K^2}^{\text{Slope match}}.
\end{aligned}
$$

Longitudinal Diffeomorphism Results