# Bottom-Up Hierarchical Image Segmentation Using Region Competition and the Mumford-Shah Functional

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## Abstract

*This paper generalizes the methods in a previous paper [10] in two ways. First, a more comprehensive analysis of the initialization problem of the Chan-Vese models is given. Second, the image segmentation method proposed in [10] is improved by applying bimodal curve evolution with region competition. The improved method maintains the advantages of the previous method. It is efficient, stable in the presence of strong noise and able to handle complicated images. It outperforms the previous method for images with weak edges. Experimental results in this paper demonstrate these improvements.*

## 1. Introduction

The Mumford-Shah model [8] is one of the most widely studied mathematical models for image segmentation. The model attempt to find image segments with all the following three properties: (1) approximate to the original image, (2) having small variations in regions, and (3) carrying short boundaries between regions. The Mumford-Shah model, especially the well-posed Mumford-Shah model, produces piecewise constant image segments.

Many methods have been proposed for the implementation of the Mumford-Shah model, such as variational methods [2], an elliptic approximation method by $\Gamma$-convergence [1], and curve evolution methods [3] [5] [12] [13].

Among all the methods, these curve evolution methods are mathematically well-founded [4] [6], and have stable implementations [7] [9]. Curve evolution methods drive one or more initialized curves, based on the minimization of the Mumford-Shah functional, to the boundaries of objects in the image. Curve evolution methods are able to handle topological changes automatically. Furthermore, since the Mumford-Shah functional combines both region and gradient information in the image, these curve evolution methods can be made robust to noise and weak edges.

Curve evolution methods, however, still have problems. First, most have initialization problems, which means different initial curves result in different segmentations for the same image. Second, these methods have difficulty with complicated images with multiple junctions. Top-down hierarchical methods [12] or multiple coupled evolving curves [13] are used to segment multiple junctions. Both top-down hierarchical methods and coupled curve evolution are time-consuming. Techniques are usually required to ensure that no pixels in the image are unsegmented or included in multiple segments.

Several methods are proposed to solve these problems, such as [10] and [5]. In [5], the authors illustrated the initialization problem for Chan-Vese models, and proposed a solution using a decoupled multiphase Chan-Vese model ands reduce the computational load. The method in [5] is hierarchically top-down, as in [12], so it still requires multiple segmentation passes for complex images. A mathematical analysis for the initialization problem of the bimodal Chan-Vese model was provided in [10], which proposed a bottom-up hierarchical method to minimize the well-posed Mumford-Shah functional. The authors also proposed a fast curve evolution method for the bimodal Chan-Vese model. The methods in [10] are efficient, robust in the presence of noise and are able to handle complicated images.

This paper extends the work in [10]. First, a more comprehensive mathematical analysis for the initialization problem of the Chan-Vese models is provided. Second, region competition, which includes region growing as a special case, is incorporated. The improved method keeps the advantages reported in [10] and outperforms the previous method for images with weak edges. The improved method bears some similarities with those reported in [11] and [14], but no *a priori* information or human interaction is necessary.

The paper is organized as follows: Background information on image segmentation using the Mumford-Shah functional is introduced in section 2. In section 3, the initialization problem for the Chan-Vese models is analyzed. The improved image segmentation method based on region competition and the Mumford-Shah functional is developed in 4. Experimental results are given and analyzed in section 5. Conclusions are provided in section 6.

# 2 Image segmentation using the Mumford Shah Functional

In this section, some background information on the Mumford-Shah functional and its implementations are introduced. The Mumford-Shah functional [8] is reviewed, followed by the introduction of the bimodal and multiphase Chan-Vese models.

## 2.1 The Mumford-Shah Model

Let $I_0$ be a function representing the image to be segmented and $I$ be a differentiable function representing the segmented image. Both $I_0$ and $I$ are defined on a planar domain $R$. Let $R_i$ be disjoint connected open subsets of $R$ with piecewise smooth boundaries and let $\Gamma$ be the union of the portions of the boundaries of $R_i$ inside $R$. Then the Mumford-Shah functional is defined as

$$E(I,\Gamma) = \mu^2 \iint_R (I - I_0)^2 dxdy + \iint_{R-\Gamma} \|\nabla I\| dxdy + \nu|\Gamma| \tag{1}$$

where $|\Gamma|$ represents the total length of $\Gamma$, and $\mu$ and $\nu$ are positive constant.

The functional in (1) is not necessarily well-posed. In most cases, a special case of (1), in which $I$ is restricted to be piecewise constant, is applied. The special case of the Mumford-Shah functional, which is well-posed, takes the following form

$$E(\Gamma) = \Sigma_i \iint_{R_i} (I_0 - mean_{R_i}(I_0))^2 dxdy + \nu|\Gamma| \tag{2}$$

Although the functional in (2) is well-posed and may have a global minimum, it is not convex and may have numerous local minima. This is the underlying reason that the Chan-Vese models, which will be introduced next, have the initialization problem.

## 2.2 The bimodal Chan-Vese Model

The Chan-Vese models are curve evolution implementations of the well-posed Mumford Shah functional (2). The bimodal Chan-Vese model [3] applies the functional (2) to bimodal images. The energy functional is a special case of the well-posed Mumford Shah model (2) with $i = 2$. By means of curve evolution, bimodal images are segmented into two parts, the background and the foreground, which can correspond to images of objects. *

Applying the Euler-Lagrange equation, this functional is minimized by solving the following PDE:

$$\frac{\partial \psi}{\partial t} = \delta_\epsilon(\psi)[\nu \cdot \kappa - (I_0 - c_1)^2 + (I_0 - c_2)^2] \tag{3}$$

where $\psi$ is a level set representation of an evolving curve $C$, which means $C = \{(x, y)|\psi(x, y) = 0\}$. $c_1$ and $c_2$ are the average values of pixels inside and outside $C$, respectively. $\kappa$ represents the curvature of the evolving curve. $\delta_\epsilon(\psi) = \epsilon/(\pi(\epsilon^2 + \psi^2))$, and $\epsilon$ is a positive constant.

It can be seen from (3) that the evolution of the curve is affected by two terms. The curvature term $\kappa$ regularizes the curve during evolution. The region term $-(I - c_1)^2 + (I - c_2)^2$ affects the motion of the curve. This term can / be interpreted as a competition between the region inside the evolving curve and the region outside the curve.

## 2.3 The Multiphase Chan-Vese Model

The bimodal Chan-Vese model is directly applicable only for bimodal images. The multiphase Chan-Vese model [13] has been proposed for complicated images. In this model, two or more coupled curves evolve simultaneously to segment images with multiple objects. Consider a four-phase Chan-Vese with energy functional (2) and $i = 4$. In the implementation, two coupled curves $\psi_1$ and $\psi_2$ evolve according to coupled Euler-Lagrange equations.

Suppose the initial curves divide the image into four regions: $R_{00} = \{\psi_1 < 0, \psi_2 < 0\}$, $R_{10} = \{\psi_1 > 0, \psi_2 < 0\}$, $R_{01} = \{\psi_1 < 0, \psi_2 > 0\}$, $R_{11} = \{\psi_1 > 0, \psi_2 > 0\}$, as shown in Fig. 2 (a). Let $c_{00}$, $c_{10}$, $c_{01}$, and $c_{11}$ be the average intensities inside $R_{00}$, $R_{10}$, $R_{01}$, $R_{11}$, respectively. The evolution of $\psi_1$ follows the PDE:

$$\begin{aligned}\frac{\partial \psi_1}{\partial t} = \delta_\epsilon(\psi_1)\{\nu\kappa_1 - ((I_0 - c_{11})^2 - (I_0 - c_{01})^2)H(\psi_2) \\ - ((I_0 - c_{10})^2 - (I_0 - c_{00})^2)(1 - H(\psi_2))\}\end{aligned} \tag{4}$$

where $\kappa_1 = \nabla \cdot (\frac{\nabla \psi_1}{|\nabla \psi_1|})$ is the curvature of $\psi_1$, and $H(\cdot)$ is the Heaviside function: $H(x) = 1$ when $x > 0$ and $H(x) = 0$ when $x < 0$ . A similar equation can be written for $\psi_2$.

It can be seen from (4) that the evolution of $\psi_1$ determines a boundary comprised of two parts: the part between $R_{00}$ and $R_{10}$ where $\psi_2 < 0$, and the part between $R_{01}$ and $R_{11}$ where $\psi_2 > 0$. The first part evolves due to region competition between $R_{00}$ and $R_{10}$. The evolution of the second part is driven by region competition between $R_{01}$ and $R_{11}$. Similar observations can be made for $\psi_2$. In this manner, the multiphase Chan-Vese model divides the image into several smaller regions and performs curve evolution based on competitions between these regions.

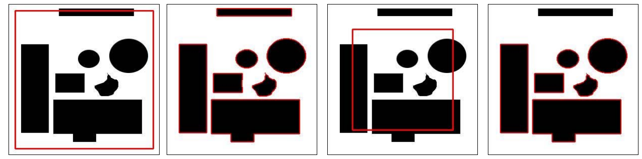
## 3. Initialization Problem of the Chan-Vese Models

An analysis is provided in this section for the initialization problem of the Chan-Vese models. Based on the background information in section 2, three observations can be made about the initialization problem.

First, initialization determines which local minimum of the energy functional (2) is achieved. Initialization in the Chan-Vese models provides the starting point for the minimization of the energy functional. Since the energy functional may have multiple local minima, and the Euler-Lagrange method is a gradient-descent method, a local minimum may be reached, for example, if the initial value is chosen to be closer to one of the local minima than the global minimum. Fig. 1 illustrates this fact. The initialization in Fig. 1(a) causes the global minimum of (2) to be found where all the objects are segmented. The initialization in Fig. 1(c), however, is closer to a local minimum of (2) than to the global minimum, and one object is not segmented as a result in Fig. 1(d) . The authors of [10] have also illustrated that the solutions to the Chan-Vese model may go through different intermediate states for different initializations even if they achieve the same local minimum.

Second, as mentioned in [10], the way the Chan-Vese model utilizes region information creates the initialization problem. In the Chan-Vese model, information from different regions are competing to evolve the curve. If initialization causes the total influence of multiple regions on the curve to be zero, nothing is segmented. Although the multiphase Chan-Vese model introduces computation between multiple small regions, initialization is still a problem without prior information about the image. The reader may refer to [10] for more details.

Third, coupling between evolving curves may magnify the effects of initialization and introduce more computational load. Fig. 2 illustrates this fact. For the initialization in (a), curve evolution based on only region information can not reach a local minimum, as shown in (d). It can be seen from (e)-(h), however, that every part of the evolving curve converges to a local minimum. Although the curvature term may finally drive the curves to a local minimum, the coupling introduces extra computation and makes the segmentation time-consuming. Therefore, it is useful to decouple the evolving curves in the multiphase Chan-Vese model for better segmentation results [5].



(a) (b) (c) (d)

**Figure 1. Different initialization in the Chan-Vese model may generate different segmentation results. (a) and (c) represent images with different initializations. (b) and (d) are the segmentation results corresponding to (a) and (c) respectively. The top rectangle is not segmented in (d).**

## 4. Image Segmentation using Region Competition and the Mumford Shah Functional

An efficient bottom-up hierarchical image segmentation method is proposed in [10] to minimize the well-posed Mumford-Shah functional (2) while avoiding the initialization problem and reducing the computational load. The method utilizes region growing to minimize the energy functional. At the beginning, each pixel in the image is taken as a region. In this way, no curves are initialized and the initialization problem is avoided. During image segmentation, two neighboring regions are merged so long as the action will decrease the energy functional. Since the well-posed Mumford-Shah model tends to segment images into piecewise constant regions, the pixels in the merged region are set to be the average intensity in the region. Such a process is repeated until no merges of regions can decrease energy. This method is shown to be efficient, robust in the presence of strong noise, and capable of handling complicated images.

The method in [10], however, can be improved. Region growing is not enough for energy minimization in some cases. Although the merges of two regions may increase the energy functional, moving their common boundaries may decrease the energy functional. Thus, region competition on the boundaries between neighboring regions can be beneficial. In fact, region growing can be taken as a special case of the region competition. This idea is similar to the ideas [11] [14] and the Chan-Vese models presented in [3] [13]. But there are differences. First, region competition in the present case results in changes between regions, not the evolution of curves. Second, region competition is performed only when region growing can not decrease the energy functional, lowering the computational burden.

Several methods may be applied for region competition, such as in [3] and [14]. The bimodal Chan-Vese model [3]

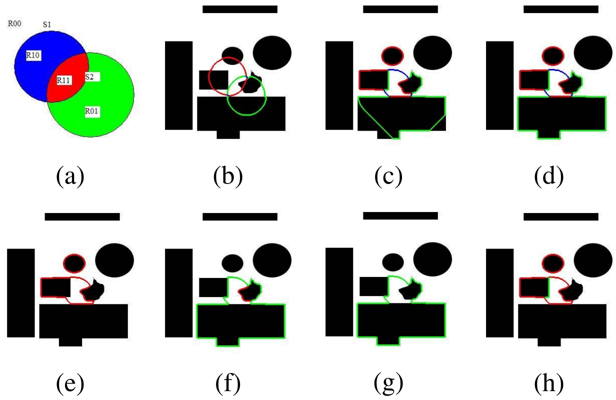



**Figure 2. Coupling between curve evolution may enlarge initialization problems. (a) Multiphase Chan-Vese model. (b) Initialization. (c) Intermediate results after 12 iterations. (d) Segmentation results. (e)-(h) show the positions of four parts of the evolving curves in (d).**

is a good choice because the geometric curve evolution in [3] can handle topological changes automatically. But [3] may be time-consuming. In the implementation, therefore, a simplified method of [3] as proposed in [10] is applied. The method evolves the boundary based on the signs of the region term $-(I-c_1)^2+(I-c_2)^2$ and $\psi$, and regularizes the boundary using Gaussian smoothing. Since it does not solve PDEs, it is very efficient.

The improved method can be seen as an extension of the multiphase Chan-Vese model when numerous initial curves are introduced so that every pixel in the image is taken as a region. It can also be viewed as a method to select the starting point for energy minimization to try to reach the the global minimum.

## 5. Experimental Results

Experimental results using the improved image segmentation method are shown in this section. The proposed method is implemented on a computer which has two Intel(R) Pentium(R) 3.2GHz CPUs, 2G bytes RAM, and runs the Red Hat Enterprise Linux operating system. The CPU times given in this paper are the sums of system CPU times and user CPU times. The system CPU time is usually very small, typically 0.01 - 0.08 second.

Fig.3 shows the comparison between the improved method and the method reported in [10]. For images with weak edges (a)(d), segmentation results of the improved method (b2)(c2)(e2) are much better than the results of the previous method. The boundaries between regions are more regularized and the results are closer to the original image. It can also be seen that both methods are very efficient.

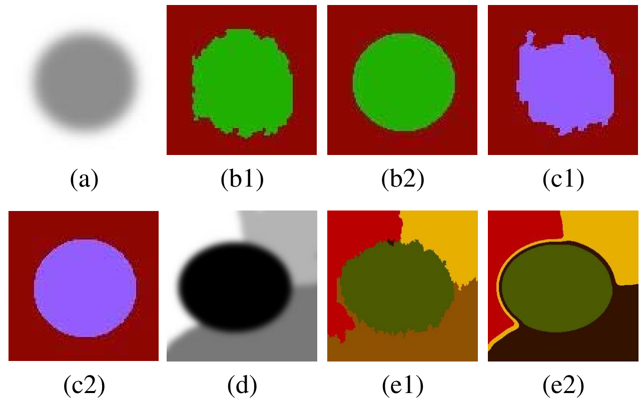



**Figure 3. Comparison of the improved model with the previous model. (a) Original image (100 * 100). (b1) Segmentation using the previous model, $\nu$ = 5000, CPU = 0.27s. (b2) Segmentation using the improved model, $\nu$ = 5000, CPU = 0.33s. (c1) Segmentation using the previous model, $\nu$ = 12000, CPU = 0.26s. (c2) Segmentation using the improved model, $\nu$ = 12000, CPU = 0.27s. (d) Original image (300 * 300). (e1) Segmentation of (d) using the previous model, $\nu = 18000$, CPU = 2.69s. (e2) Segmentation of (d) using the improved model, $\nu = 18000$, CPU = 3.52s.**

Segmentation results for more complicated images are shown in Fig. 4. (a2) illustrates that the method is stable in the presence of large noise in (a1). (b2)(c2)(d2)(g2) are the segmentation results of (b1)(c1)(d1)(g1), respectively. These results demonstrate that the proposed method is able to handle complicated images and is very efficient. (e2)(e3)(f2)(f3) show the influence of the parameter $\nu$. With $\nu$ increasing, fewer objects are segmented. The proper selection of $\nu$ for a specific image is a topic under research.

## 6. Summary

This paper generalizes the results reported in [10]. A more comprehensive analysis is provided for the Chan-Vese models, and an improved image segmentation method is proposed using region competition and the Mumford-Shah functional. The proposed method is shown to be efficient, robust in the presence of noise, and able to handle complicated images. Furthermore, it generates better results than the previous method for images with weak edges with only a minor increase in computational burden. Future work will be focused on the dynamic selection of the parameter $\nu$ and the parallel implementation of the proposed method.

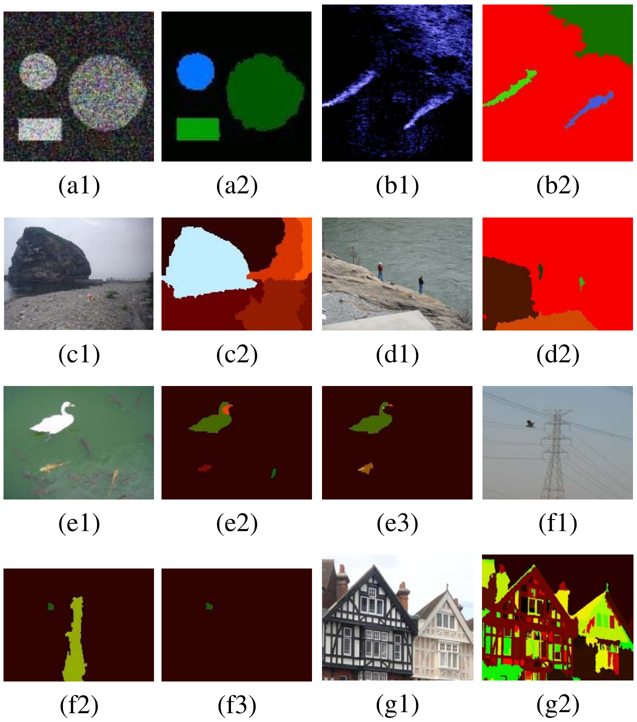



**Figure 4. Experimental results for real images. (a1)-(g1) are original images. The sizes are: (a1) (128 * 128), (b1) (255 * 266), (c1) (200 * 150), (d1) (300 * 225), (e1) (200 * 150), (f1) (200 * 150), (g1) (698 * 581). Other images are segmentation results for (a1)-(d1). Corresponding parameters are: (a2) $\nu$ = 1200, CPU = 0.82s. (b2) $\nu$ = 7000, CPU = 4.61s. (c2) $\nu$ = 2500, CPU = 2.23s. (d2) $\nu$ = 10000, CPU = 2.64s. (e2) $\nu$ = 600, CPU = 1.82s.(e3) $\nu$ = 700, CPU = 1.96s. (f2) $\nu$ = 600, CPU = 0.93s.(f3) $\nu$ = 1500, CPU = 0.91s. (g2) $\nu$ = 3000, CPU = 27.16s.**

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