1 Elliptic Harmonics

In the complex notation $z(u)$ is represented as a series of complex exponentials.

$$z(s) = \sum_{n=-\inf}^{\inf} z_n e^{\frac{2\pi jns}{L}} = z_0 + \sum_{n=1}^{\inf}(z_n e^{\frac{2\pi jns}{L}} + z_{-n} e^{-\frac{2\pi jns}{L}})$$

where the complex coefficient z_n can be expressed in polar notation, i.e.

$$z_n = r_n e^{j\psi_n} \ ,$$

with $r_n \in R, r_n \geq 0$, and $\psi \in R$. The terms $e^{\frac{2\pi jns}{L}}$ describe rotations as a funct. of arclength s. We have demonstrated that the terms $(z_n e^{\frac{2\pi jns}{L}} + z_{-n} e^{-\frac{2\pi jns}{L}})$ form ellipses that are traversed n times while traversing the figure from 0 to L. Remember the demonstration with pairs of phasors of different length $|z_n|$ rotating clockwise and counterclockwise, and resulting vector being the sum of the clockwise phasor $z_n = \{a, b\}$ and the counterclockwise phasor $z_{-n} = \{c, d\}$.

1.1 Normalization in object and parameter space

The normalization proposed by *Kuhl and Giardina* is based on the ellipse defined by the 1$^{\text{st}}$ order Fourier descriptors and is carried out both in object and in parameter space. Normalization in object space effects the curve's position, orientation and size, while that in parameter space applies to the curve parametrization behind it. After normalization in object space the center of the 1$^{\text{st}}$ order ellipse of a normalized contour concurs with the coordinate origin, its main axis overlaps with the x-axis of the coordinate system has the length of 1. In parameter space the starting point of the parametrization is moved to a standard position defined by the crossing of the 1$^{\text{st}}$ order ellipse and its main axis.

1.1.1 Dependence on starting point

To make the descriptors independent on the starting point of the parametrization, this can be shifted to a standard position, e.g. to the tip of the ellipse defined by the 1$^{\text{st}}$ order Fourier descriptors. This can be thought of as a rotation in parameter space U given by the unit circle. The transformation is defined by

$$z_n \mid^V = z_n e^{jn\theta} \ , \tag{1}$$

where the notation $|^V$ marks the coefficients resulting from shifting by angle θ.

1.1.2 Dependence on rotational position

In the complex notation, rotation in object space by angle ψ is simply a multiplication by $e^- j\psi$. Applying it to 1 immediately reveals the coefficients of the rotated object.

$$z_n \mid^R = z_n e^{j\psi} \tag{2}$$

To achieve a standardized position of the curve, it is rotated in a way that its first ellipse's main axis matches the horizontal (real) coordinate axis.

1.1.3 Scale dependence

Scaling the objects by factor α leads to multiplying its coefficients by the same factor:

$$z_n \,|^{S} = \alpha z_n \tag{3}$$

The scaling factor α is usually set to normalize the half major axis to unity, meaning

$$\alpha = \frac{1}{|z_1| + |z_{-1}|} = \frac{1}{r_1 + r_{-1}} \tag{4}$$

1.1.4 Invariant Fourier descriptors

Ignoring z_0, that is setting $z_0 \,|^{T} = 0$, achieves translation invariance. Summing up all standardizations; the invariant coefficients are denoted $\tilde{z}_n$:

$$z_n \,|^{V,R,S,T} = \tilde{z}_n \quad = \quad z_n \frac{e^{j(n\theta - \psi)}}{r_1 + r_{-1}} \tag{5}$$

$$\tilde{z}_0 \quad = \quad 0 \tag{6}$$

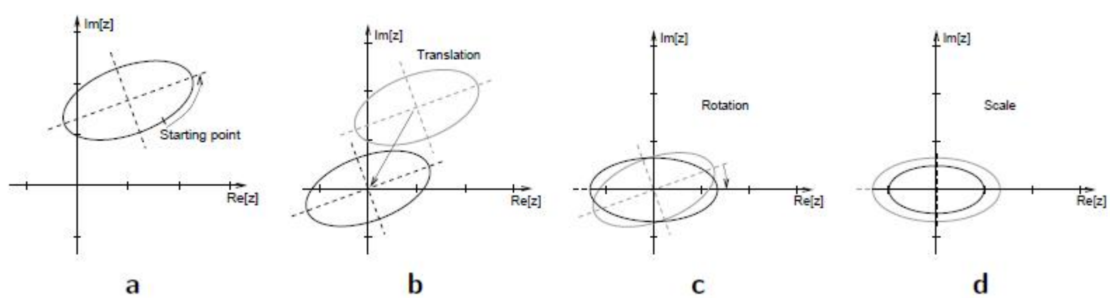

Figure 1: Normalization steps of Fourier coefficients; shifting of the starting point to the tip of the ellipse (**a**), moving the center of gravity to the coordinate origin (**b**), rotating the main axis of the ellipse to the real axis (**c**), and finally scaling the half major axis to unity (**d**).

1.2 Relationship to real valued notation

There is a close relationship between the complex and real notation (see the document *Kelemen-EllipticHarmonicsOnly.pdf*), i.e. they can be converted into each other.
In the complex notation of Fourier coefficients real and imaginary parts of z_n correspond to the x and y coordinates

$$\begin{pmatrix} x \\ y \end{pmatrix}_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \begin{pmatrix} \sin \frac{2\pi n s}{L} \\ \cos \frac{2\pi n s}{L} \end{pmatrix},$$

where the real valued coefficients a_n, b_n, c_n, and d_n are defined as follows.

$$
\begin{aligned}
a_n &= \mathrm{Re} z_n + \mathrm{Re} z_{-n} \\
b_n &= -\mathrm{Im} z_n + \mathrm{Im} z_{-n} \\
c_n &= \mathrm{Im} z_n + \mathrm{Im} z_{-n} \\
d_n &= \mathrm{Re} z_n - \mathrm{Re} z_{-n}
\end{aligned}
$$

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Calculation of Coefficients (ctd.)

1

equation (1) does not have a closed form solution

trick: $\frac{d}{ds}$: $$Z'(s) = \sum_{m=-\infty}^{\infty} \frac{2\pi j m}{L} \cdot z_m \cdot e^{\frac{2\pi j m s}{L}} \quad \Big| \cdot e^{-2\pi j n \frac{s}{L}}$$

$$\Downarrow$$

$$Z_n = \frac{1}{2\pi j n} \int_0^L Z'(s)\, e^{-2\pi j n \frac{s}{L}}\, ds$$

our contour is piecewise constant: 

$\Rightarrow$ interval p: $\frac{\Delta Z_p}{|\Delta Z_p|} = \frac{\Delta Z_p}{|\Delta s|}$ = constant!

constant speed

$$\Rightarrow \underline{\underline{Z_n}} = \frac{1}{2\pi j n} \sum_{k=0}^{M-1} \int_{s_k}^{s_{k+1}} Z'(s)\, e^{-2\pi j n \frac{s}{L}}\, ds$$

$$= \frac{1}{2\pi j n} \sum_{k=0}^{M-1} Z'(k) \int_{s_k}^{s_{k+1}} e^{-2\pi j n \frac{s}{L}}\, ds$$

$$= \frac{1}{2\pi j n} \sum_{k=0}^{M-1} Z'(k) \left[\frac{-L}{2\pi j n} e^{-2\pi j n \frac{s}{L}} \right]_{s_k}^{s_{k+1}}$$

$$= \frac{L}{4\pi^2 n^2} \sum_{k=0}^{M-1} \left(Z'(k)\, e^{-2\pi j n \frac{s}{L}} \Big|_{s_k}^{s_{k+1}} \right)$$

$$= \frac{L}{4\pi^2 n^2} \sum_{k=0}^{M-1} \frac{\Delta Z_p}{|\Delta s|} \cdot \left(e^{-2\pi j n \frac{s_{k+1}}{L}} - e^{-2\pi j n \frac{s_k}{L}} \right) \quad (2)$$

equation (2) can be solved by integration of piecewise constant vectors along contour!

Calculation of Coefficients

(see also Kuhl & Giardina 1982)

$$Z(s) = \sum_{m=-\infty}^{\infty} Z_m e^{2\pi \frac{ims}{L}}$$

$$Z(s) \cdot e^{-\frac{2\pi jns}{L}} = \sum_{m=-\infty}^{\infty} Z_m e^{2\pi j(m-n)\frac{s}{L}}$$

$$\int_0^L Z(s) e^{-2\pi jn\frac{s}{L}} ds = \sum_{m=-\infty}^{\infty} Z_m \underbrace{\int_0^L e^{2\pi j(m-n)\frac{s}{L}}}_{\circledast \begin{cases} L & \text{if } m=n \\ \emptyset & \text{otherwise} \end{cases}}$$

$$= Z_n \cdot L$$

$$\Rightarrow Z_n = \frac{1}{L} \int_0^L Z(s) e^{-\frac{2\pi jns}{L}} ds \qquad (1)$$

$\circledast$ $m=n: \int_0^L e^0 \, ds = L$ ✓

$$m \neq n: \quad -\frac{L}{2\pi j(m-n)} e^{2\pi j(m-n)\frac{s}{L}} \Big|_0^L =$$

$$-\frac{L}{2\pi j(m-n)} (\underbrace{e^{2\pi j(m-n)} - 1}_{1-1}) = \emptyset \quad ✓$$