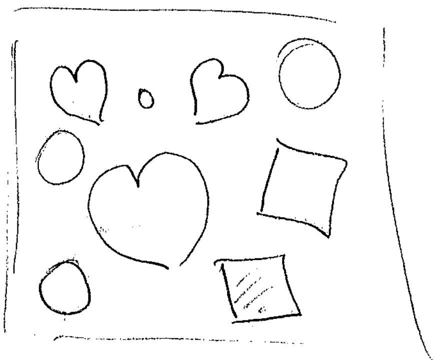
02/25/2010

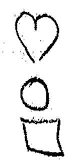
...age Analysis: Fourier Harmonics

Guido Gerig (1)

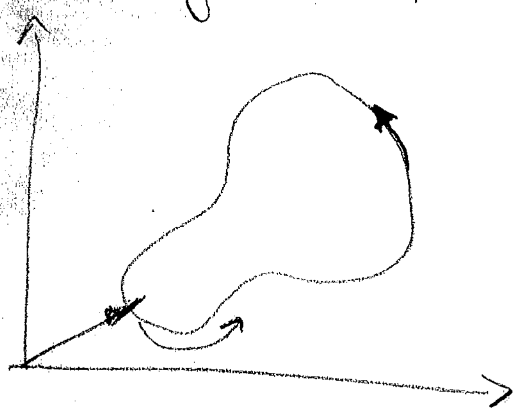


- Simplified description of object (small set of parameters)
- "Gestalt" parameters: Robust w.r.t. small deformation, continuous
- Invariant under:
  - scaling
  - translation
  - rotation
- Either area or contour

Equivalence classes:



Boundary Description:



$$\underline{x}(s) = \begin{pmatrix} x(s) \\ y(s) \end{pmatrix} \qquad s: 0,..,L$$

$\Rightarrow \underline{x}(s)$ is periodic with s

s measured on original contour

Invariance: Two contours are equivalent $\underline{x}_1 \sim \underline{x}_2$, if $\underline{x}_1 = \alpha \cdot R \cdot \underline{x}_2 + t$ | scaling, rotation, translation

Equivalence: | Reflexivity: $\underline{x}_1 \sim \underline{x}_1$

Symmetry: $\underline{x}_1 \sim \underline{x}_2 \Rightarrow \underline{x}_2 \sim \underline{x}_1$

Transitivity: $\underline{x}_1 \sim \underline{x}_2$ and $\underline{x}_2 \sim \underline{x}_3 \Rightarrow \underline{x}_1 \sim \underline{x}_3$

group property of linear

02/25/2010 (2)

Reflexivity: $\alpha = 1,\ \varphi = 0,\ t = 0$

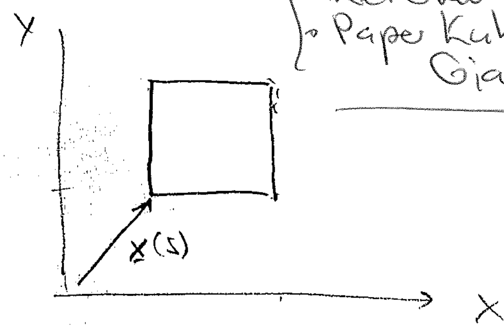
Symmetry: $\frac{1}{\alpha}(\underline{x}_1 - t) = R\, x_2 \Rightarrow x_2 = R^{-1}\left(\frac{1}{\alpha}(\underline{x}_1 - t)\right)$

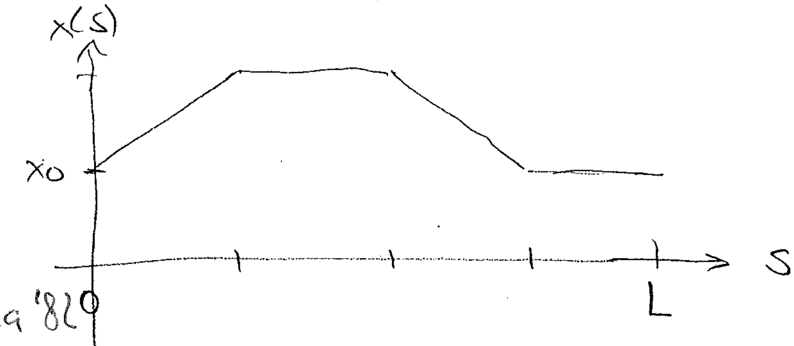
Transitivity: . . .

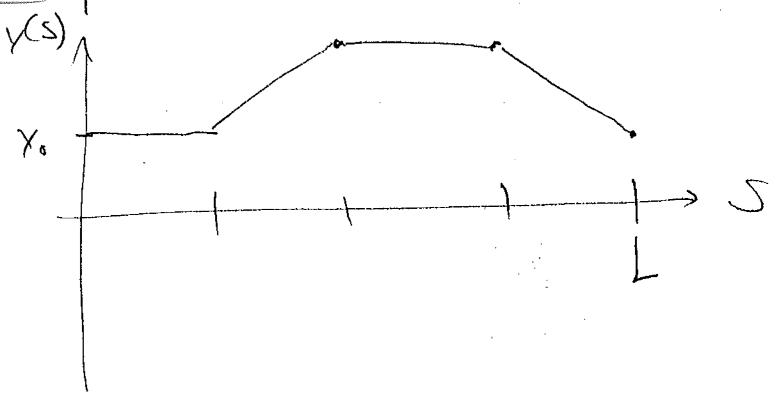
Fourier Descriptors:

Material: Chapters
- Brechbühler
- Kelemen
- Paper Kuhl & Giardina '82

Example:







idea: $\left.\begin{matrix} x(s) \\ y(s) \end{matrix}\right\}$ Fourier Series

Kuhl & Giardina (1982):

$$x(s) = A_0 + \sum_{n=1}^{\infty} a_n \cos 2\pi (n/L) s + b_n \sin 2\pi (n/L) s$$

$$y(s) = B_0 + \sum_{n=1}^{\infty} c_n \cos \qquad d_n \sin$$

$$A_0 = \frac{1}{L} \int_0^L x(s)\, ds \qquad B_0 = \frac{1}{L} \int_0^L y(s)\, ds$$

$$a_n = \frac{2}{L} \int_0^L x(s) \cos \frac{2\pi n s}{L}\, ds \qquad b_n = \dots$$

$$c_n = \frac{2}{L} \int_0^L y(s) \cos \frac{2\pi n s}{L}\, ds \qquad c_n = \dots$$

$\boxed{\text{more elegant: complex notation: } z(s) = x(s) + j\, y(s)} = |z| \cdot e^{j\phi}$

02/25/2010 ③

$$\boxed{Z(s) = \sum_{n=-\infty}^{\infty} Z_n e^{2\pi j \left(\frac{n}{L}\right) s}}$$

Series of complex exponential functions

$$= Z_0 + \sum_{n=1}^{\infty} \left(Z_n e^{2\pi j \left(\frac{n}{L}\right) s} + Z_{-n} e^{-2\pi j \left(\frac{n}{L}\right) s} \right)$$

$$\left[Z_0 = \frac{1}{L} \int_0^L Z(s)\, ds = \frac{1}{L} \int_0^L (x(s) + i\, y(s))\, ds \right.$$

$$\left. \begin{aligned} Re(Z_0) &= \frac{\sum x}{L} = \bar{x} \\ Im(Z_0) &= \frac{\sum y}{L} = \bar{y} \end{aligned} \right\} \text{Center of gravity of contour}$$

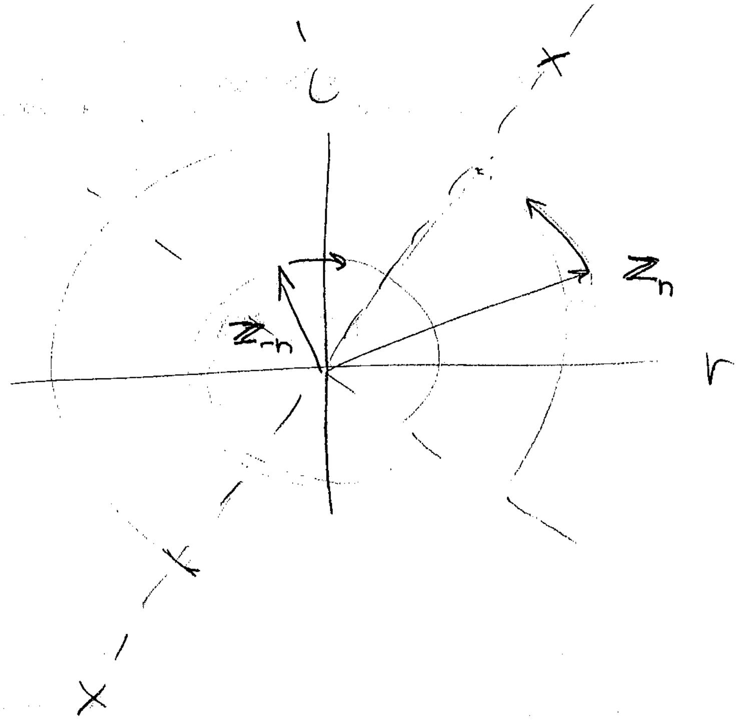
$$\underset{n \neq 0}{Z_n} = \frac{2}{L} \int_0^L Z(s)\, e^{j \frac{2\pi}{L} n s}\, ds$$

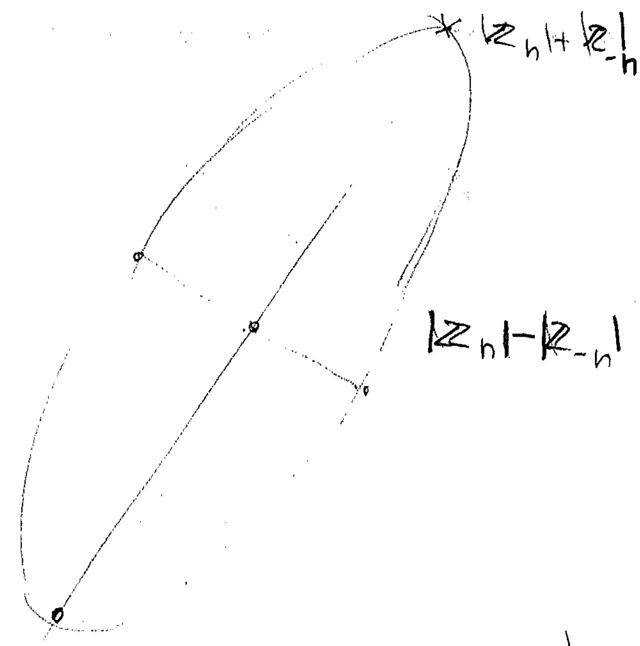
$$Z_{-n} = \frac{2}{L} \int_0^L \underbrace{Z(s)}_{x(s) + i\, y(s)} \underbrace{e^{-j \frac{2\pi}{L} n s}}_{\text{rotator in complex space}}\, ds$$

$Z_n^* \neq Z_{-n}$ (

conj. complex

$$Z(s) = \underset{\substack{\uparrow \\ \text{Center of gravity} \\ \text{translation}}}{Z_0} + \sum_{n=1}^{\infty} \underbrace{\left(Z_n e^{2\pi j \left(\frac{n}{L}\right) s} + Z_{-n} e^{-2\pi j \left(\frac{n}{L}\right) s} \right)}_{?}$$





show!

harmonic contributions: ellipses

$n=1$: $2\pi j \frac{s}{L}$: ellipse surrounded once from $0..L$

$n=2$: $2\pi j \frac{2s}{L}$: ellipse surrounded twice from $0..L$

$n=n$: ...

$$\Rightarrow z(s) = z_0 + \underbrace{\sum_{n=1}^{\infty} \text{ellipses}}_{\text{superposition of ellipses}}$$

clock work of rotating pointers

$\Rightarrow$ Java Demo's

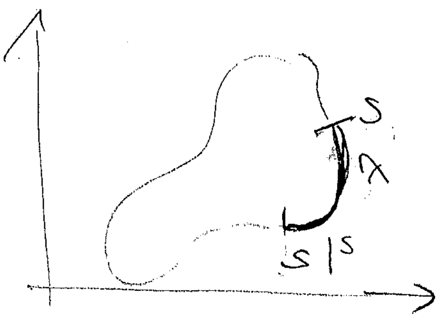
Invariance

02/25/2010 (5)

given $z(s)$

- Translation: Set z_0 to ϕ

- Starting Point: (series exp. varies with selection of starting point)



$$s = s|^{\vee} + \lambda$$

$|^{\vee}$ marks quantities that result from shifting by λ

$$\tilde{z}(s) = z(s|^{\vee} + \lambda) = \sum z_n e^{\frac{2\pi j n (s|^{\vee} + \lambda)}{L}} =$$

$$z|^{\vee}(s|^{\vee}) = \sum_{n=-\infty}^{\infty} \underbrace{z_n e^{\frac{2\pi j n s|^{\vee}}{L}} \cdot e^{\frac{2\pi j n \lambda}{L}}}_{z_n|^{\vee}}$$

$$\Rightarrow z_n|^{\vee} = z_n \cdot e^{jn\theta} \qquad \theta = \frac{2\pi\lambda}{L}$$

$\Rightarrow$ with θ: we can move the starting point to an arbitrary position along the contour.

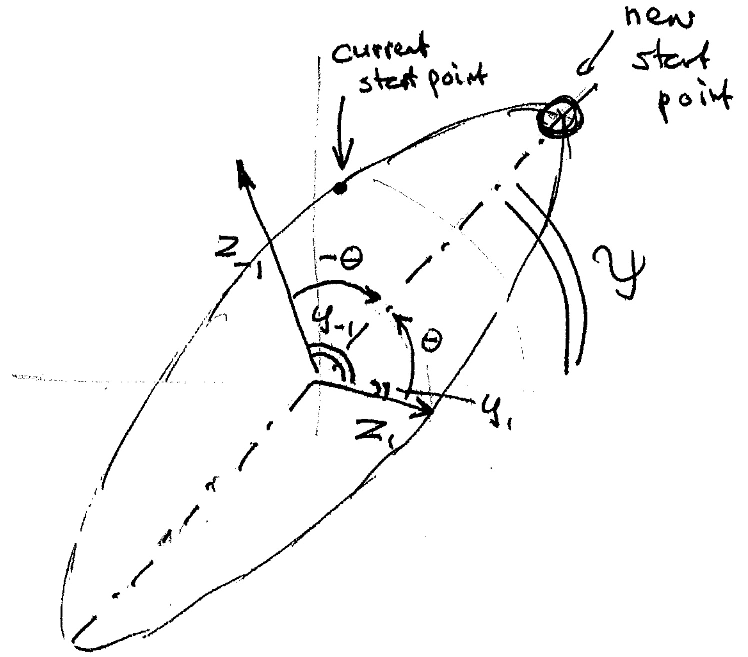
good choice?

e.g.: vertex of first order ellipse

CS 7960 AIP | Feb. 25, 2010

Fourier Harmonics II

(6a)



Start point

$Z_{-1} : \varphi_{-1}$: $Z_1 = r_1 e^{i\varphi_1}$

$Z_1 : \varphi_1$: $Z_2 = r_2 e^{i\varphi_2}$

$$\varphi_1 + \theta = \varphi_{-1} - \theta = \psi \quad \text{(angle of ellipse)}$$

$$2\theta = \varphi_{-1} - \varphi_1$$

$$\boxed{\theta = \frac{\varphi_{-1} - \varphi_1}{2}} : \text{"shift" angle}$$

$$\boxed{\psi = \varphi_1 + \theta = \frac{\varphi_1 + \varphi_{-1}}{2}}$$

rotation of ellipse

$$\Rightarrow \quad \mathrm{elli}_1\big|^{v} = Z_1 e^{i(\varphi_1 + \theta)} + Z_{-1} e^{i(\varphi_{-1} - \theta)}$$

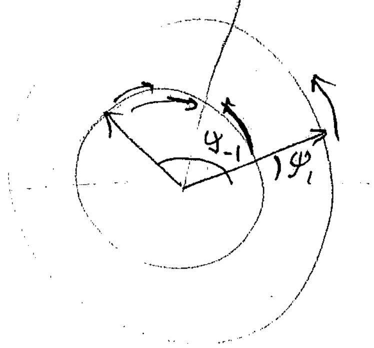
$$= Z_1 e^{i(\psi)} + Z_{-1} e^{i(\psi)}$$

$$\overset{\text{vertex}}{=} (Z_1 + Z_{-1}) \cdot e^{i(\psi)}$$

how to find ?

02/25/2010

6b



vertex: both pointers in phase!

$$\varphi_1 + \theta = \varphi_{-1} - \theta = \varphi$$

$$\theta = \frac{\varphi_{-1} - \varphi_1}{2} \quad \text{necessary shift}$$

$$\Rightarrow \varphi = \frac{\varphi_1 + \varphi_{-1}}{2} \quad \text{position of main axis}$$

(shifting starting point = rotation in parameter space, the unit circle)

do it: $$elli_1|^{\varphi} = Z_1 e^{j(\varphi_1 + \theta)} + Z_{-1} e^{j(\varphi_{-1} - \theta)}$$

$$\overset{\text{vertex}}{=} (Z_1 + Z_{-1}) e^{j\varphi}$$

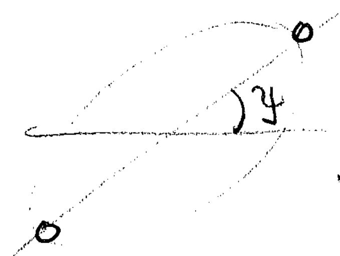
$\Rightarrow$ first ellipse: calculate θ through current starting point

rotate coefficients by $\boxed{Z_n|^{\varphi} = Z_n e^{jn\theta}}$

- **Rotation:** rotate coordinate system by φ or object by $-\varphi$

$$Z|^R(s) = Z(s) \cdot e^{-j\varphi} = \sum_{n=-\infty}^{\infty} Z_n e^{-j\varphi} \cdot e^{\frac{2\pi j n s}{L}}$$

$$\boxed{Z_n|^R = Z_n e^{-j\varphi}}$$



why did we use φ? Main axis of first ellipse becomes new real axis !

Problem: ellipse has two vertices → ambiguity!

important! all coefficients multiplied by the same phasor

• Scaling

02/25/2010 7

$$z|^S(s) = \alpha \, z(s) = \sum_{n=-\infty}^{\infty} (\alpha z_n) e^{\frac{2\pi j n s}{L}}$$

$$z_n|^S = \alpha z_n \qquad | \text{ scale factor } \alpha$$

normalization, e.g. long axis of first ellipsoid becomes 1 : $\boxed{\alpha = \frac{1}{|z_1| + |z_{-1}|}}$

• all together:

calculate $z(s) = z_0 + \sum$ ellipses ...

first ellipse $elli_1$ gives ψ, θ, $z_1 + z_{-1}$

$$\Rightarrow \boxed{z_n|^{V,R,S,T} = \tilde{z}_n = z_n \frac{e^{j \cdot (n\theta - \psi)}}{|z_1| + |z_{-1}|}}$$

$$\tilde{z}_0 = 0$$

$\Rightarrow$ object representations become independent of T, R, S, start point. | $\tilde{z}_n$ = descriptor, shape feature vector

• truncation

$$\breve{z}(s) = \sum_{n=-N}^{N} z_n e^{\frac{2\pi j n s}{L}}$$

$$f(s) = z(s) - \breve{z}(s) = \sum_{|n|>N} z_n e^{\frac{2\pi j n s}{L}}$$

mean square error ϵ_2

$$\epsilon_2^2 = \int_0^L |f(s)|^2 ds$$

energy in img space

$$= \int_0^L |\underbrace{z(s) - \breve{z}(s)}_{\sum_{|n|>N} z_n e^{\frac{2\pi}{L} i n s}}|^2 ds = \int_0^L |f(s)|^2 ds$$

$$= \dots = L \sum_{|n|>N} |z_n|^2 = L \sum_{|n|>N} r_n^2 = \epsilon_2^2$$

Sentence of Parseval

energy in frequency space

|| mse between parametrized curves via sum of coefficients ||

$$\underbrace{\int_0^L |z(s)|^2 ds}_{\text{energy in image space}} = \underbrace{L \cdot \sum_{k=\infty}^{\infty} |z_n|^2}_{\text{energy in frequency space (coefficients)}}$$

Classification of Objects

model object g: descriptor $\{Z_n[g]\}$

unknown object u: descriptor $\{Z_n[u]\}$

classification distance:

$$D[g] = \sum_{n=-N}^{N} \left| \tilde{Z}_n[g] - Z_n[u] \right|^2$$

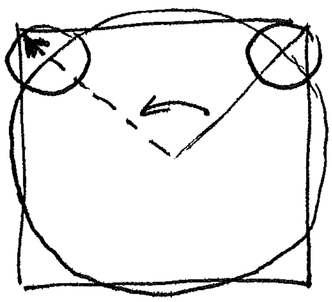
if several models:

- calculate distances $D[g_1]$
  $D[g_2]$
  $\vdots$
  $D[g_K]$
- assign new object u to class g_i with smallest distance:

$$i = \underset{g_i}{\operatorname{argmin}} \{ D[g_i], \; g_i = \text{objects} \}$$

Quizz: Approximation of square

Fourier $\longrightarrow$ sharp cusps / corners?



which orders will be necessary?

- order $\boxed{1}$: circle

$\Rightarrow$ ~~order 4~~: NO

order 1: we move $+90°$

order x: also $+90°$ for corner correction

or: $\underbrace{360° + 90°}$ = $450°$

full cyle: $4 \cdot 450° = 1800$

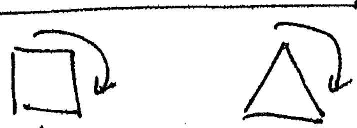
$= 5 \cdot 360° \Rightarrow \boxed{5}$ order

or: $\underbrace{-270°}$

full cycle: $4 \cdot -270° = -1080°$

$= 3 \cdot -360° \Rightarrow \boxed{3}$ order

In general: coefficients for rotational symmetry?



$k=4$ $k=3$

Repetition: $\frac{2\pi}{k}$

rule: $Z\left(s+\frac{L}{k}\right) = e^{\frac{2\pi j}{k}} Z(s)$

shift by $\frac{L}{k}$ $\Rightarrow$ rotation in parameter space

$$\Rightarrow Z\left(s+\frac{L}{k}\right) - e^{\frac{2\pi j}{k}} Z(s) = 0$$

(coefficients the same after object rotation with $\frac{2\pi}{k}$)

$$\Rightarrow \sum Z_n e^{\frac{2\pi j n s}{L}} e^{\frac{2\pi j n}{k}} - \sum Z_n e^{\frac{2\pi j n s}{L}} e^{\frac{2\pi j}{k}} = 0$$

$$\Rightarrow \sum Z_n e^{\frac{2\pi j n s}{L}} \cdot e^{\frac{2\pi j}{k}} \left(\underbrace{e^{2\pi j \left(\frac{n-1}{k}\right)}}_{e^{2\pi j \left(\frac{n-1}{k}\right)} = 1} - 1 \right) = 0$$

$$\frac{n-1}{k} \in \mathbb{N}_1 = \{1, 2, \ldots\}$$

(similar in other direction ↻)