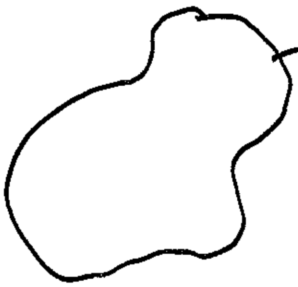
Active Contours II: Level-set evolution ①

Material:
- Caselles, Kimmel, Sapiro, IJCV 1997
- Sethian: math.berkeley.edu/~sethian/Movies or /Explanations
- Yushkevich et al.: Insight-SNAP

- Explicit parametrization of active contours:
  - issues of parametrization, re-parametr.
  - " " calculation of derivatives v_s and v_{ss}
  - extension to 3D ("active meshes") requires sophisticated update scheme (see Sethian et.a.)
  - topology is fixed

- Alternative:
  - embed curve into higher order function → curve is represented as zero level of level set $u(x)$
  - apply PDE to $u(x,t)$

Let us build an active contour by curve evolution:

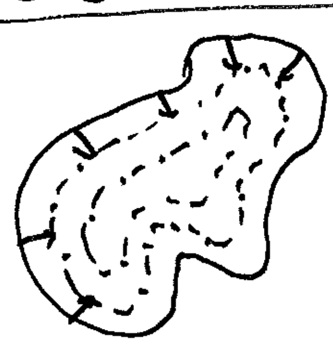


$c(s,t) : [0..1]$

a) $\boxed{\dfrac{\partial c}{\partial t} = \beta \cdot \vec{N}}$

β : speed

$\vec{N}$: unit inward normal



Curve moves inwards (or outwards if β is negative) with speed β

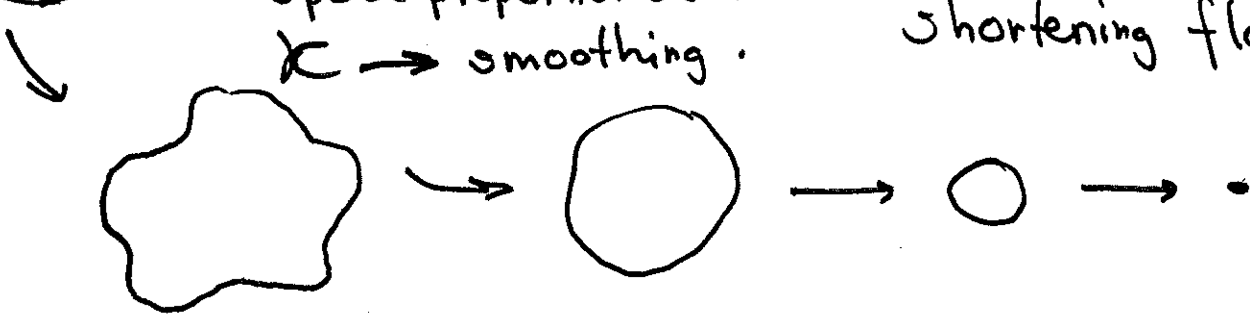
b) $\boxed{\dfrac{\partial c}{\partial t} = \varkappa \vec{N}}$

$\varkappa$: Euclidean curvature

known: – fastest way to reduce length of contour (Caselles, Sapiro, ...)

– Euclidean curve shortening flow

Curve moves in direction of sign of curvature (convex, concave) and speed proportional to $\varkappa \rightarrow$ smoothing.



combine a) and b):

$$\boxed{\frac{\partial C}{\partial t} = (\beta + \kappa)\vec{N}}$$

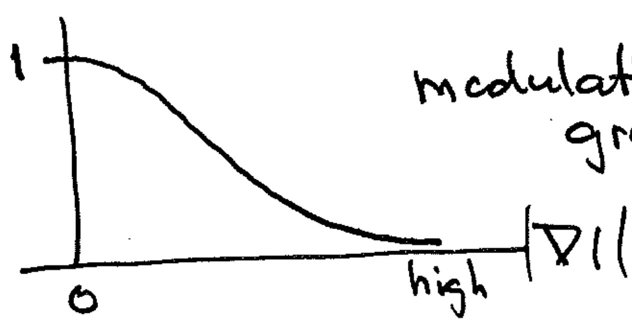
propagation and smoothing

c) what is left: image force term so that curve stops at boundary

$$\boxed{\frac{\partial C}{\partial t} = \underbrace{g(I)}\,(\beta + \kappa)\vec{N}}$$

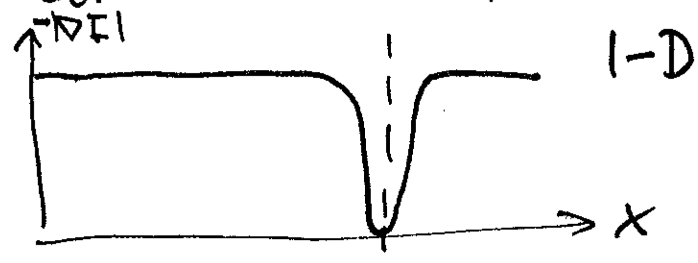
$$g(I) = g(|\nabla I|)$$

see anisotropic diffusion (Perona & Malik)

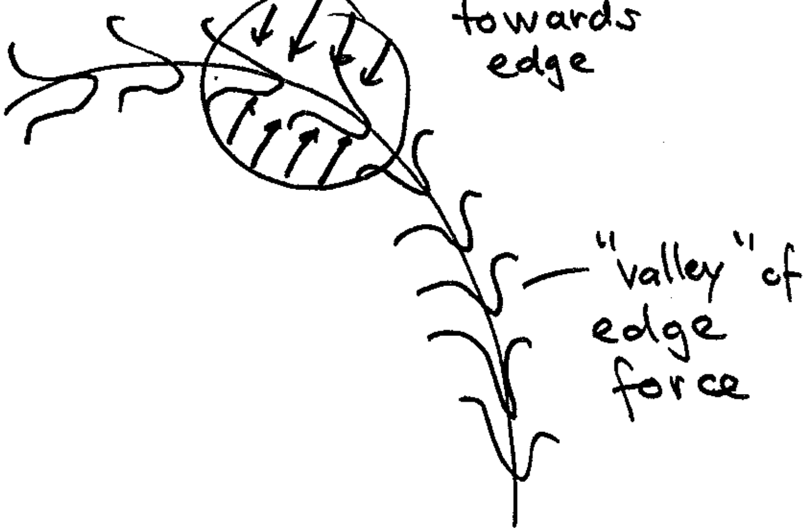


modulating term; large gradient magnitude → to 0 → halt evolution

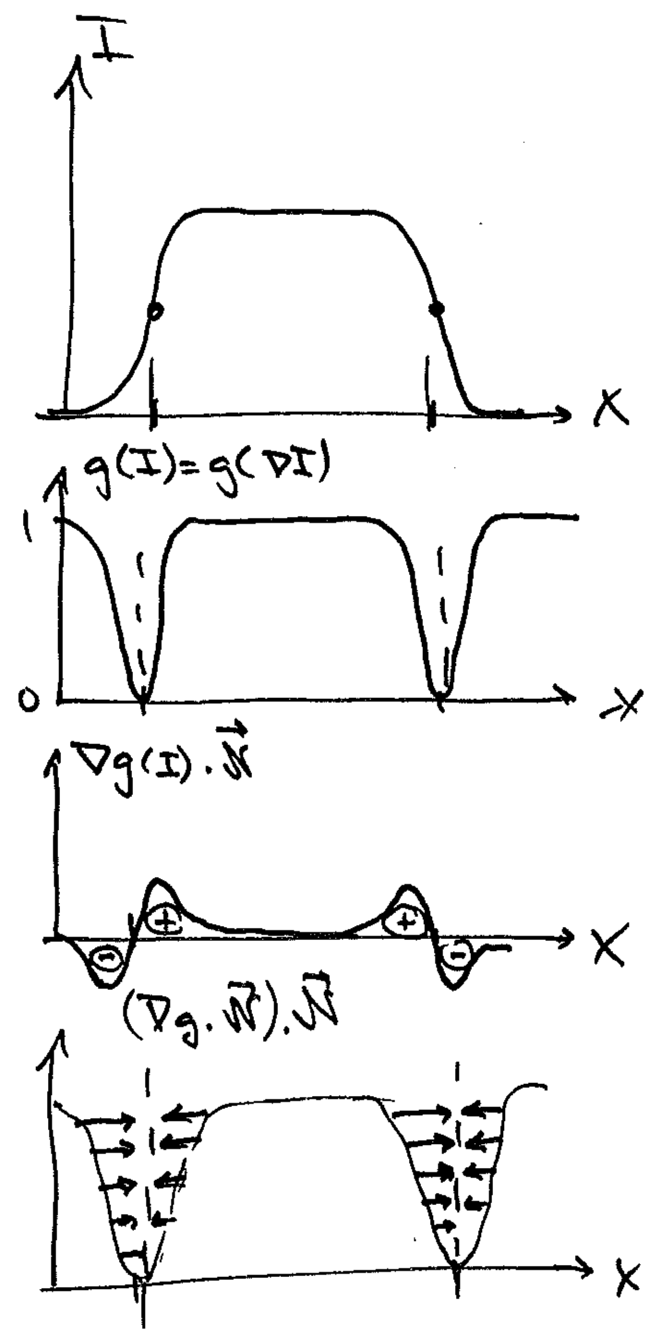
d) more creativity: "gradient advection" term (curve should not only stop near boundary but "lock in" at location of gradient magnitude max)



$$\boxed{\frac{\partial C}{\partial t} = g(I)(\beta + \kappa)\vec{N} - \underbrace{(\nabla g(I) \cdot \vec{N})\vec{N}}_{\text{advection term}}}$$



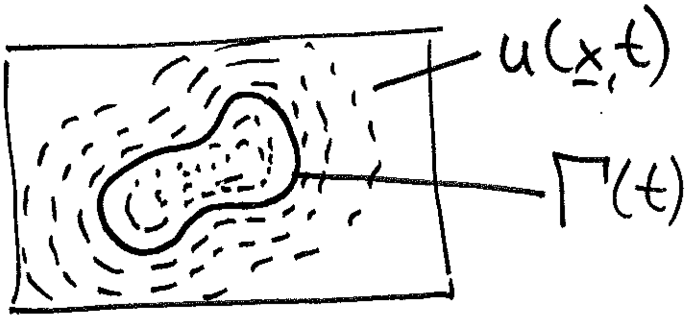
(see equation (20) in Caselles et. al.)

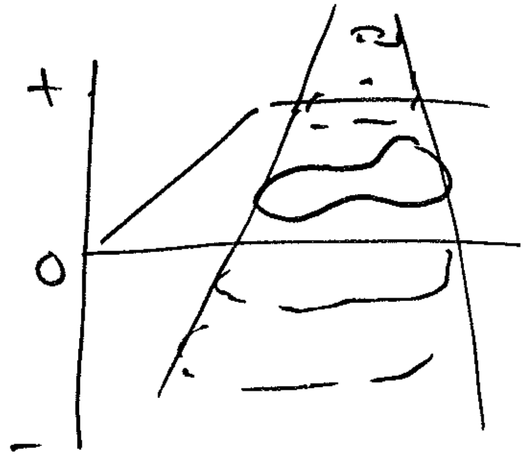


Embedding $c(s)$ into level set u

embedding function $u(\underline{x},t)$

embedded curve $c(s,t)$: $u(\Gamma,t)=0$
(zero level)





height function

let's do the math:

$$u(\Gamma,t)=0$$

$\Gamma\in\mathbb{R}^2$; 0-level set

$$\frac{d}{dt}:\ \nabla u\cdot\Gamma_t+u_t=0$$

$$\frac{\nabla u}{|\nabla u|}=-\vec{N}$$

chain rule $\nabla u\cdot\Gamma_t=-u_t$

$$C_t=\beta\vec{N}=\beta\cdot-\frac{\nabla u}{|\nabla u|}$$

plug in our curve evolution

$$-\nabla u\cdot\beta\cdot\frac{\nabla u}{|\nabla u|}=-u_t$$

$$\frac{\nabla u\cdot\nabla u}{|\nabla u|}=|\nabla u|$$

$$\boxed{\beta|\nabla u|=u_t}$$

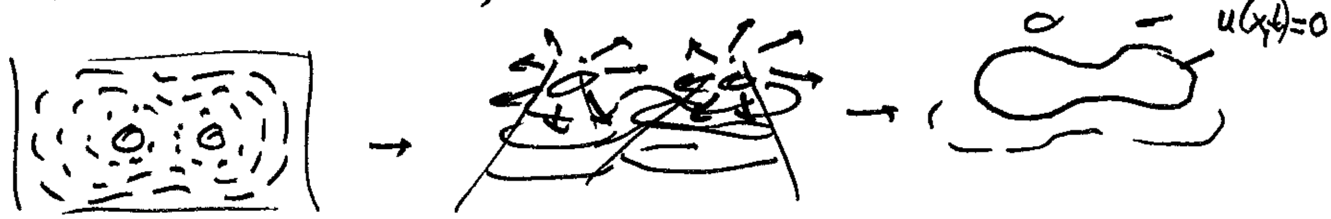
Punch line:

Evolving embedding function $u(\underline{x},t)$ as $u_t = \beta\,|\nabla u|$, it's embedded curve $u(\Gamma,t)=0$ evolves like

$$\frac{\partial c}{\partial t} = \beta \vec{N} \quad .$$

Practical implication: Instead of explicitely solving for a parametrized curve, we can apply a PDE to the embedding function $\longrightarrow$ parametrization-free, implicit

Advantages: • Topological changes of $c(t)$ are handled automatically via $u(x,t)$.



- New scheme allows split and merge.
- proper numerical alg. available to solve PDE (applied math)
- no explicit parametrization, contour is a level set !!!

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put it together:

(eq. (15) Caselles)

$$\frac{\partial u}{\partial t} = g(I)(\beta + \kappa)|\nabla u|$$

$$= g(I)\left(|\nabla u| \underbrace{\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)}_{\kappa} + \beta\, g(I)|\nabla u|\right)$$

means that each of the level sets of u is evolving according to:

$$C_t = (\beta + \kappa)\vec{N} \quad \text{(Osher \& Sethian 1988)}$$

why interesting? PDE might offer simpler and more stable solutions than explicit parametrization of curves represented as polygons. Extension to 3D is straightforward.

more insight:

a) $$\frac{\partial u}{\partial t} = \left(\operatorname{div} \frac{\nabla u}{|\nabla u|}\right) \cdot |\nabla u|$$

$$= \underbrace{\nabla \cdot \left(\frac{\nabla u}{|\nabla u|}\right) \cdot |\nabla u|}_{\text{MCF}}$$

- Euclidean shortening flow,
- mean curvature flow MCF

$$\Longrightarrow \Longrightarrow \quad \frac{\partial L}{\partial t} = L_{vv}$$ (see chapter 6.5 FEV)

$$\frac{\partial u}{\partial t} = u_{vv}$$ (second derivative along isophote)

b) Gaussian diffusion: $|\nabla u| = \text{constant}$ (not locally adaptive)

$$\Rightarrow \frac{\partial u}{\partial t} = \operatorname{div} \nabla u = \nabla^2 u = \Delta u$$

Laplacian

Important contribution of Caselles et. al.

Interesting: Relationship between energy-based active contour minimization and level-set evolution:

$$E(C) = \alpha \int_0^1 |C'(q)|^2 dq + \beta \int_0^1 |C''(q)|^2 dq - \lambda \int_0^1 |\nabla I(C(q))| \, dq$$

$|v_s|^2$ $|v_{ss}|^2$ $P(v)$

$$\Downarrow$$

$\beta = 0$

$$\Downarrow$$

$|-\nabla I| \rightarrow g(|\nabla I|)$

$$\Downarrow$$

embedded curve: (eq. (13) Caselles)

$$\frac{\partial C}{\partial t} = g(I)\underbrace{\kappa \vec{N}}_{\text{MCF}} - \underbrace{(\nabla g \cdot \vec{N})\vec{N}}_{\text{advection term}} \quad (13)$$

$$\Downarrow$$

embedding space

$$\frac{\partial u}{\partial t} = g(I)|\nabla u|\kappa + \nabla g(I) \cdot \nabla u) \quad (14)$$