Normalized Graph cuts

by

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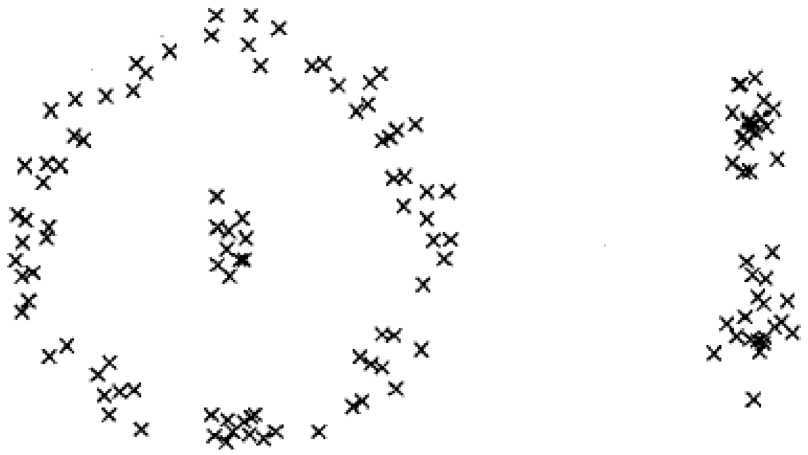
Image segmentation

- Image segmentation is a grouping technique used for image. It is a way of dividing an image into different regions that possess similar properties such as intensity, texture, color etc.

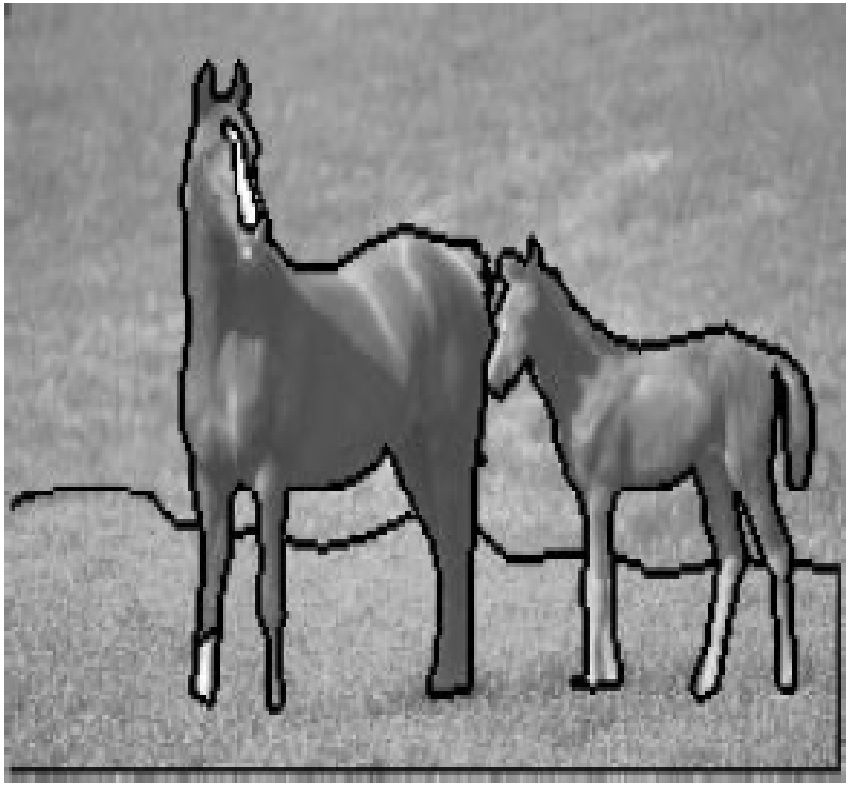


Perceptual Grouping

Organizing image primitives into higher level primitives thus extracting the global impression of the image rather than focusing on local features in the image data

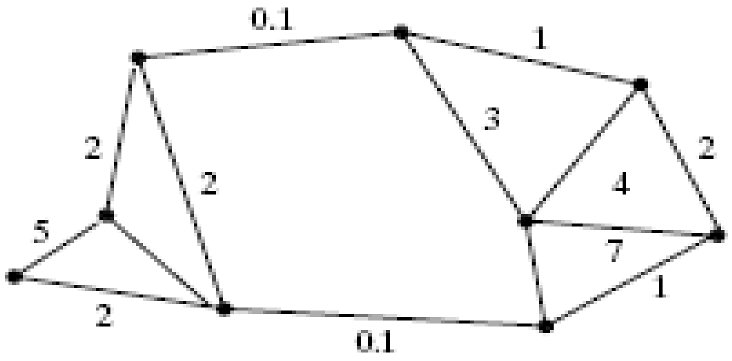
Original image

Segmented image



Graph terminology*

- A graph is a set of nodes V and edges E that connect various nodes; G={V,E}
- A weighted graph is the one in which weight is associated with each edge.
- A connected graph is the one where every pair of nodes is connected.



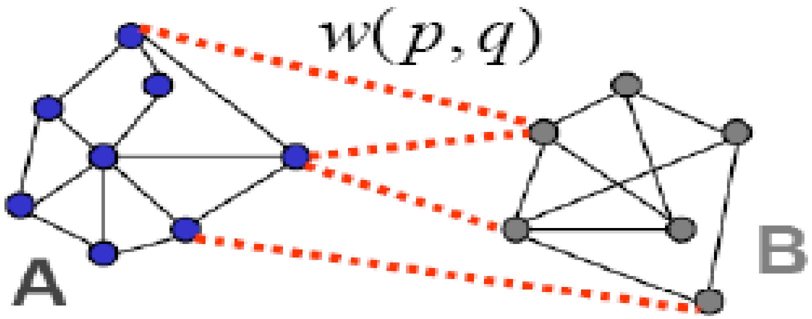
Weighted Graph

*'Segmentation by Graph-Theoritic Clustering' Forsyth, Ponce, Computer Vision book, pp:317-326

Graph cuts

- In grouping, a weighted graph is split into disjoint sets (groups) where by some measure the similarity within a group is high and that across the group is low.
- A graph-cut is a grouping technique in which the degree of dissimilarity between these two groups is computed as the total weight of edges removed between these 2 pieces.

Minimum Graph cuts image segmentation



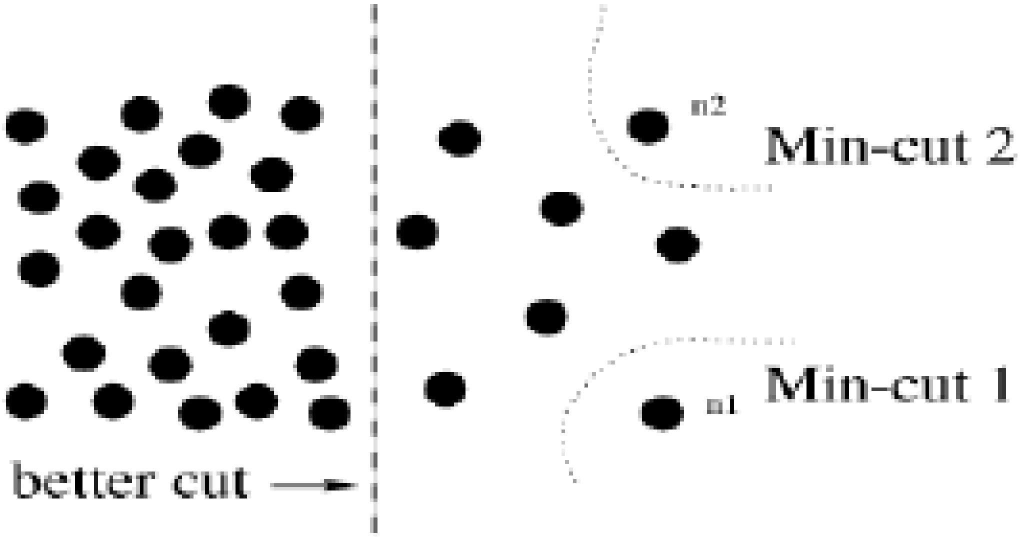
$$cut(A,B) = \sum_{p \in A, q \in B} w(p,q)$$

- Criteria: By minimizing this cut value, one can optimally bi-partition the graph and achieve good segmentation.

$$\min \ cut(A,B)$$

Drawbacks of Graph cuts

- The minimum cut criteria occasionally supports cutting isolated nodes in the graph due to the small values achieved by partitioning such nodes.



Minimum Normalized Cut Image Segmentation

- Normalized cut [1,2] computes the cut cost as a fraction of the total edge connections to all the nodes in the graph.

$$Ncut(A,B) = \frac{cut(A,B)}{assoc(A,V)} + \frac{cut(A,B)}{assoc(B,V)}$$

where $assoc(A,V) = \sum_{u \in A, t \in V} w(u,t)$

Advantage: Being an unbiased measure, the *Ncut* value with respect to the isolated nodes will be of a large percentage compared to the total connection from small set to all other nodes.

Computation of optimum partition using ${}_{min}N_{cut}$

$$Ncut(A,B) = \frac{cut(A,B)}{asso(A,V)} + \frac{cut(B,A)}{asso(B,V)} = \frac{\sum_{(\boldsymbol{x}_i>0,\boldsymbol{x}_j<0)} -w_{ij}\boldsymbol{x}_i\boldsymbol{x}_j}{\sum_{\boldsymbol{x}_i>0} \boldsymbol{d}_i} + \frac{\sum_{(\boldsymbol{x}_i<0,\boldsymbol{x}_j>0)} -w_{ij}\boldsymbol{x}_i\boldsymbol{x}_j}{\sum_{\boldsymbol{x}_i<0} \boldsymbol{d}_i}$$

where*, x* is an *N* dimensional indicator vector such that x_i=1 if 'i' is in A and -1 if 'i' is in B. $d(i) = \sum_j w(i,j)$, total , total connection from node i to all other nodes.

Let D be an N x N diagonal matrix, W be an N x N symmetric matrix with W(i,j) = w_{ij} , $k = \frac{\sum_{x_i>0} d_i}{\sum_i d_i}$, and 1 be an N x 1 vector of all ones.

$$4[Ncut(\boldsymbol{x})] = \frac{(\mathbf{1}+\boldsymbol{x})^T(\mathbf{D}-\mathbf{W})(\mathbf{1}+\boldsymbol{x})}{k\mathbf{1}^T\mathbf{D}\mathbf{1}} + \frac{(\mathbf{1}-\boldsymbol{x})^T(\mathbf{D}-\mathbf{W})(\mathbf{1}-\boldsymbol{x})}{(1-k)\mathbf{1}^T\mathbf{D}\mathbf{1}}$$

Derivations

Derivations

Let us consider there are 3 nodes such that
2 nodes in group A
1 node in group B

So,
$$x = \begin{bmatrix} \overset{A}{1} & \overset{A}{1} & \overset{B}{-1} \end{bmatrix}$$

$$\sum_{(x_i>0,\, x_j<0)} -w_{ij}x_i x_j \Rightarrow -w_{13}x_1x_3 = w_{13}$$
$$-w_{23}x_2x_3 = w_{23}$$

$$4N_{cut}\,(1^{st}\text{ part numerator}) = 4(w_{13}+w_{23}) \quad \text{——— (1)}$$

$$(1+x)^T(D-W)(1+x) \Rightarrow [2\ \ 2\ \ 0]\begin{bmatrix} w_{12}+w_{13} & -w_{12} & -w_{13} \\ -w_{21} & w_{21}+w_{23} & -w_{23} \\ -w_{31} & -w_{32} & w_{31}+w_{32} \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

$$= 4[w_{13}+w_{23}] \quad \text{——— (2)}$$

(1) = (2)

$$k = \frac{\sum_{x_i>0} d_i}{\sum_i d_i} = \frac{w_{11}+w_{21}+w_{22}+w_{13}+w_{23}+w_{12}}{w_{11}+w_{12}+w_{13}+w_{21}+w_{22}+w_{23}+w_{31}+w_{32}+w_{33}}$$

$$k1^TD1 = \frac{w_{11}+w_{21}+w_{22}+w_{13}+w_{23}+w_{12}}{\;//\;}\left(\;\; // \;\;\right)$$

$$= w_{11}+w_{21}+w_{22}+w_{13}+w_{23}+w_{12} = \sum_{x_i>0} d_i \ \left[\text{denominator part}\right]$$

Computation of optimum partition using ${}_{min}N_{cut}$

- Letting y=(1+x)-b(1-x),
- Solution:

$$min_{\boldsymbol{x}} Ncut(\boldsymbol{x}) = min_{\boldsymbol{y}} \frac{\boldsymbol{y}^T(\boldsymbol{D}-\boldsymbol{W})\boldsymbol{y}}{\boldsymbol{y}^T\boldsymbol{D}\boldsymbol{y}}$$

with the condition $y^T D 1 = 0$

$$D = \begin{bmatrix} \sum_j w(1,j) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sum_j w(N,j) \end{bmatrix}, \quad W = \begin{bmatrix} w(1,1) & \cdots & w(1,N) \\ \vdots & \ddots & \vdots \\ w(N,1) & \cdots & w(N,N) \end{bmatrix}$$

Derivations

$$y^T D 1 = \sum_{x_i>0} d_i - b \sum_{x_i<0} d_i = 0$$

$$b = \frac{k}{1-k} = \frac{\omega_{11}+\omega_{12}+\omega_{21}+\omega_{22}+\omega_{13}+\omega_{23}}{\omega_{31}+\omega_{32}+\omega_{33}}$$

$$\sum_{x_i>0} d_i = \omega_{11}+\omega_{12}+\omega_{13}+\omega_{22}+\omega_{23}+\omega_{21}$$

$$\sum_{x_i<0} d_i = \omega_{31}+\omega_{32}+\omega_{33}$$

$$\therefore\ y^T D 1 = \omega_{11}+\omega_{12}+\omega_{13}+\omega_{22}+\omega_{23}+\omega_{21} - \left(\frac{\omega_{11}+\omega_{12}+\omega_{21}+\omega_{22}+\omega_{13}+\omega_{23}}{\omega_{31}+\omega_{32}+\omega_{33}}\right)(\omega_{31}+\omega_{32}+\omega_{33})$$

$$= 0$$

$$(D-\omega)y = \lambda D y \quad \text{[Generalized Eigen System]}$$

$$(D-\omega)D^{-1/2}z = \lambda D D^{-1/2} z \quad \left(\text{since } y = D^{-1/2} z\right)$$

$$D^{-1/2}(D-\omega)D^{-1/2}z = \lambda D^{-1/2} D D^{-1/2} z$$

$$D^{-1/2}(D-\omega)D^{-1/2}z = \lambda z \quad \text{[Standard Eigen System]}$$

Computation of optimum partition using ${}_{min}N_{cut}$

- Minimizing the equation using standard eigensystem.

$$D^{-\frac{1}{2}}(D-W)D^{-\frac{1}{2}}z = \lambda z, \quad where\ z = D^{\frac{1}{2}}y$$

- Smallest eigen vector, $z_0=D^{1/2}1$ with eigen value 0
- Second smallest eigen vector, $z_1^T z_0 = y_1^T D1$ =0
- Thus using the Rayleigh quotient [3], the second smallest eigen vector turns out to be the real valued solution to the normalized cut problem.
- one can subdivide the existing graph using the eigen vector with the next smallest eigen value.

Summary of Normalized cut grouping algorithm

- Given a set of features, construct a weighted graph by computing weight on each edge and then placing the data into W and D.
- Solve $(D-W)x=\lambda Dx$ for eigen vectors with the smallest eigenvalues.
- Use the eigen vector corresponding to the second smallest eigenvalue to bipartition the graph into two groups.
- Recursively repartition the segmented parts if necessary.

K-way normalized cut

$$Ncut_k = \frac{cut(A_1, V - A_1)}{assoc(A_1, V)} + \frac{cut(A_2, V - A_2)}{assoc(A_2, V)} + \ldots + \frac{cut(A_k, V - A_k)}{assoc(A_k, V)}$$

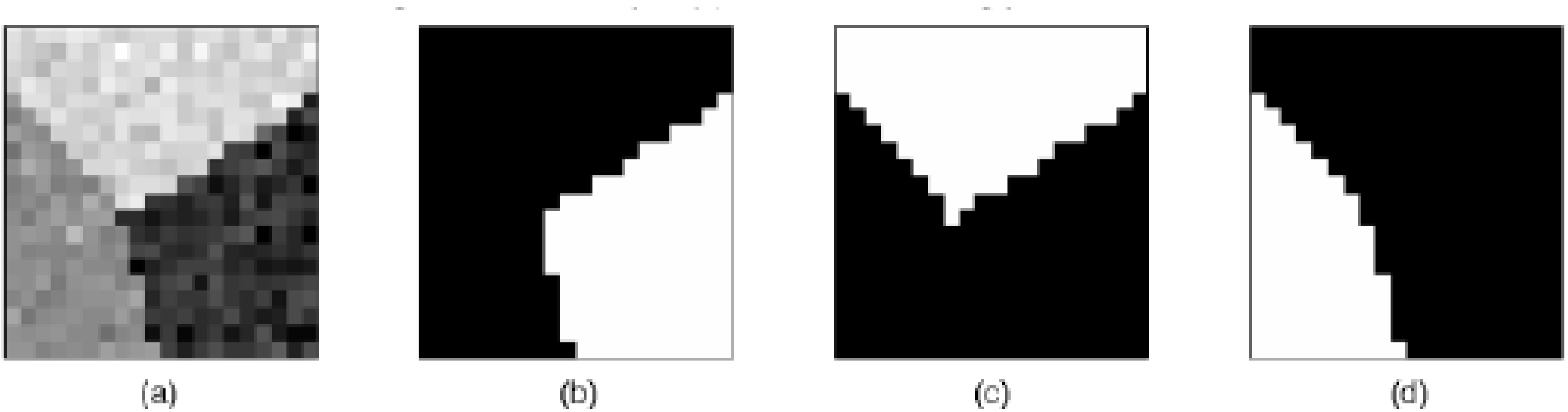
Definition of edge weight

$$w_{ij} = e^{\frac{-\|\boldsymbol{F}_{(i)} - \boldsymbol{F}_{(j)}\|_2}{\sigma_I}} * \begin{cases} e^{\frac{-\|\boldsymbol{X}_{(i)} - \boldsymbol{X}_{(j)}\|_2}{\sigma_X}} & \text{if } \|\boldsymbol{X}(i) - \boldsymbol{X}(j)\|_2 < r \\ 0 & \text{otherwise} \end{cases}$$

X_i=Spatial location of node i

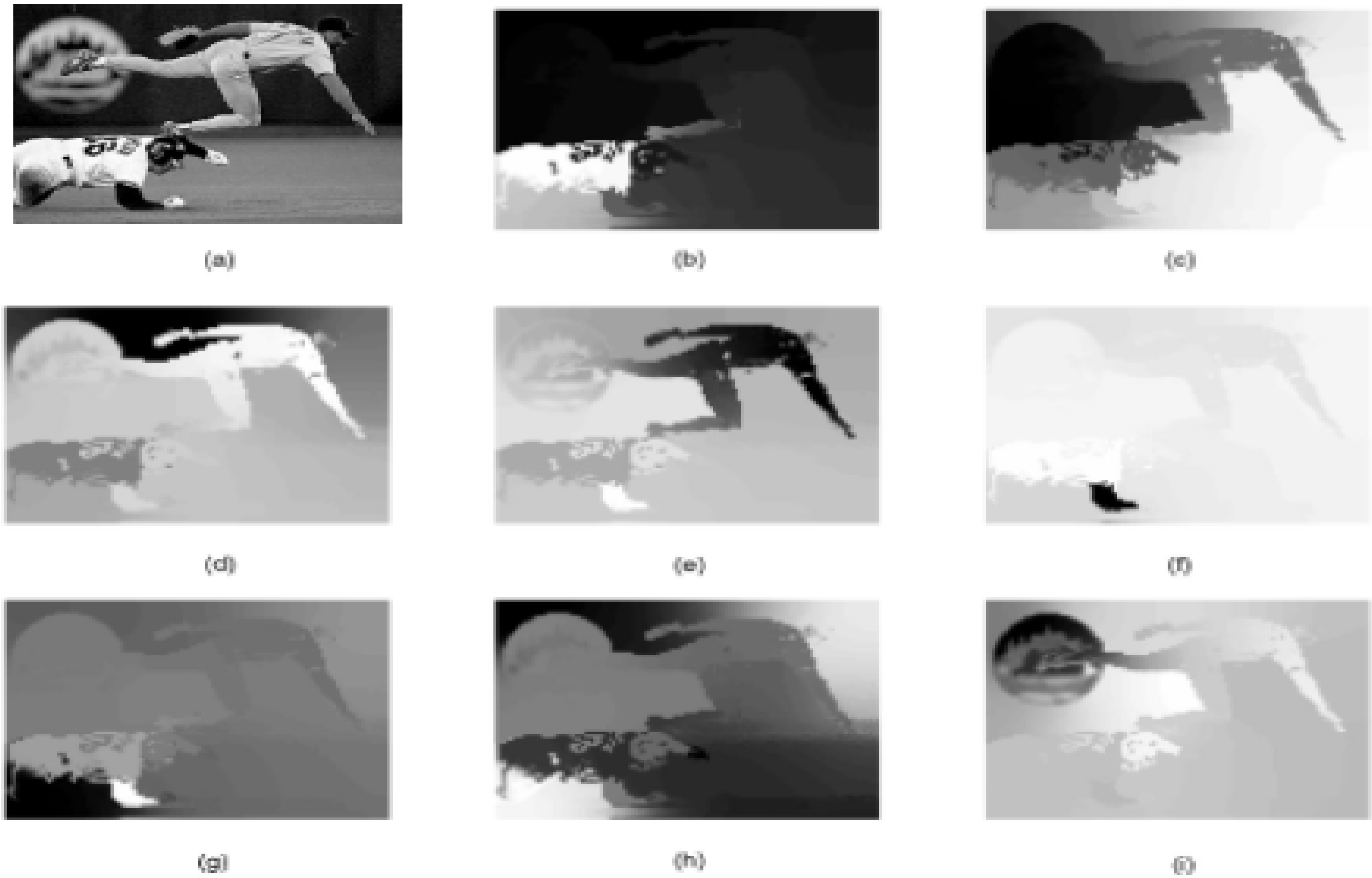
$F_{(i)}$=feature vector based on intensity, color or texture at node i

Results



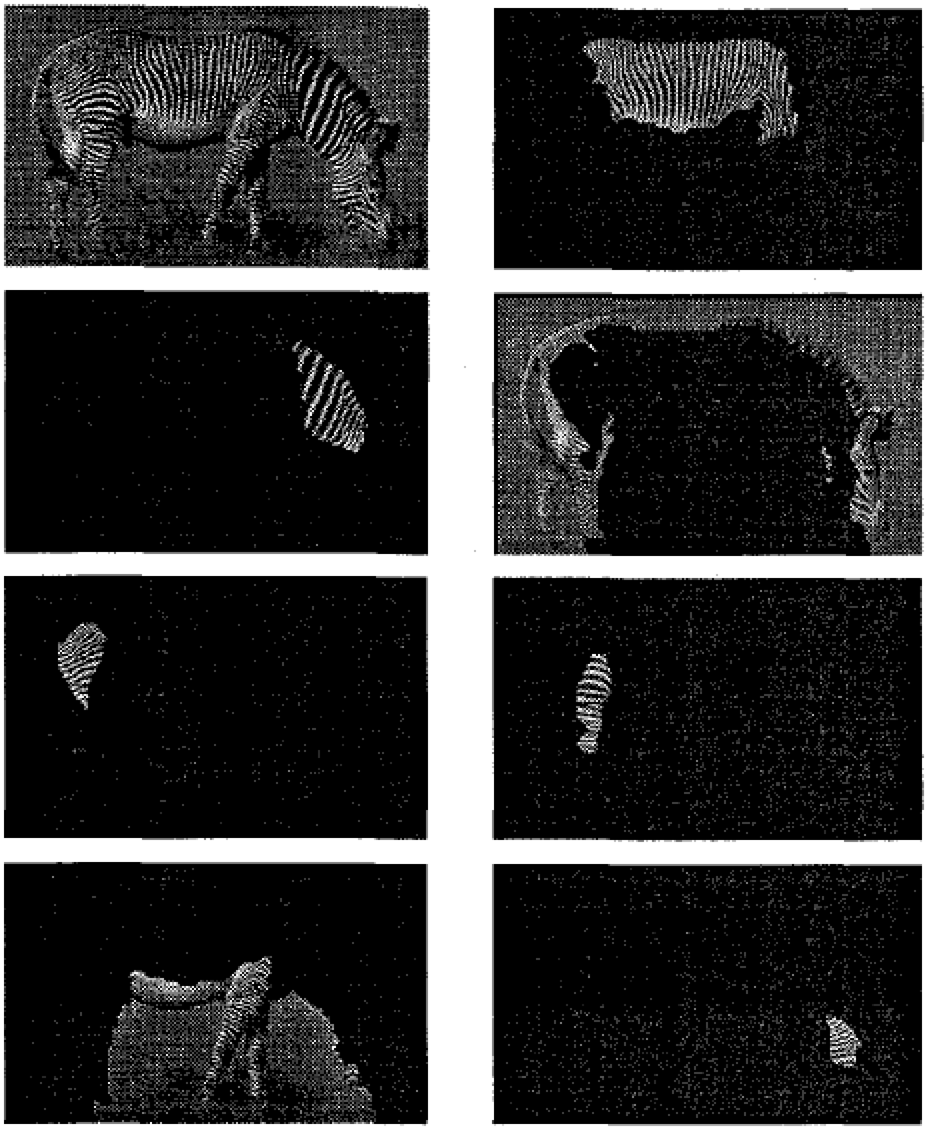
(a) A synthetic image showing three image patches forming a junction and Gaussian noise with σ = 0.1 is added. (b-d) top three components of the partition

Results



a) A 80x100 baseball scene, with feature as image intensity, (b-h) components of partition with Ncut value less than 0.04, Parameter setting: σ_i=0.01, σ_X=4 and r=5

Results



The first image is a zebra image that undergoes $_{Ncuts}$ texture segmentation with measure as orientation variant. The remaining images show the components of partition.

References:

[1] J. Shi and J. Malik, Normalized Cuts and Image Segmentation, Proc. IEEE Conf. Computer Vision and Pattern Recognition, pp. 731-737, 1997.

[2] J. Shi and J. Malik, Normalized Cuts and Image Segmentation, IEEE Transactions On Pattern Analysis And Machine Intelligence, Vol. 22, No. 8, August 2000

[3] G.H. Golub and C.F. Van Loan, Matrix Computations. John Hopkins Press, 1989.

Acknowledgements:

- Prof. Gerig
- Wei Liu

Image partitioning is to be down from the big picture downwards, like a painter who starts with major regions and then filling in the details

Thanks!