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Original Paper on Elliptic Fourier Features:
Frank P. Kuhl and Charles Giardina, Elliptic Fourier Features of a Closed Contour,
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4

Parametrization of closed curves and surfaces

Parametrically deformable models give rise to the question of obtaining parametrical descriptions of given pixel or voxel based object contours or surfaces, as they usually result from manual segmentation.

In the following the term "*parametrization*" will be used in two different senses. On the one hand, the process which maps one parameter value (s) to each point of a curve or two parameter values (s,t) to each point of a surface is called *curve* or *surface parametrization*.

$$\boldsymbol{v}(s) = \begin{pmatrix} v_1(s) \\ v_2(s) \end{pmatrix} = \begin{pmatrix} x(s) \\ y(s) \end{pmatrix} \tag{4.1}$$

in 2-D, or

$$\boldsymbol{v}(s,t) = \begin{pmatrix} v_1(s,t) \\ v_2(s,t) \\ v_3(s,t) \end{pmatrix} = \begin{pmatrix} x(s,t) \\ y(s,t) \\ z(s,t) \end{pmatrix} \tag{4.2}$$

in 3-D. On the other hand, these mappings can be used to give a mathematical representation of the contour by the coordinate functions in $\boldsymbol{v}(s)$ and $\boldsymbol{v}(s,t)$. As it has been shown in the previous chapter, the coordinate functions also depend on parameters (descriptors), e.g., the weights of some basis functions. Confusingly, the procedure to compute these descriptors is also called *parametrization*. To make a clear distinction, in the following *curve* or *surface parametrization* will refer to the mapping procedure and simply *parametrization* to the computation of shape descriptors.

The computation of the *parametrization* of a given binary object usually obtained from manual segmentation is a three-stage process. In the first stage, the contour or surface of the object is converted into an abstract data structure called *chain* or *crack code* in 2-D and *cuberille notion* in 3-D describing the shape as a set of line segments or a set of rectangular patches. Considering only the case of simply connected objects, in the second stage, based on these notations, curves are mapped on the unit circle Ω_2, while surfaces on the unit sphere Ω_3. Arc length parametrization guarantees constant speed for $\boldsymbol{v}(s)$ along the curve, while the equivalent criteria of area preservation for surfaces can be at most approximately fulfilled. In the third stage a mathematical representation of the object is

computed by deriving the parameters (descriptors) of the chosen functions, e.g., Fourier basis or superquadrics.

4.1 Fourier coefficients of a closed curve

Kuhl and Giardina in [Kuhl and Giardina 1982] presented an algorithm to efficiently compute the Fourier coefficients of a 2-D contour. Their elegant approach does not require integration or the use of fast Fourier transform techniques. The algorithm takes the chain code of the contour as its input, which is easily obtained from a pixel-based image. The resulting Fourier descriptors can be made invariant with rotation, dilatation and translation of the contour, and also with the starting point of the parametrization on the contour, without losing information about the shape.

4.1.1 Obtaining the code of a contour

The *chain* code first described by Freeman approximates a contour by a sequence of piecewise linear fits that consist of eight standardized line segments. The code of a contour is then the chain V of length K, $V = a_1 a_2 a_3 \ldots a_K$, where each link a_i is an integer between 0 and 7 oriented in the direction $(\pi/4)a_i$ and of length 1 or $\sqrt{2}$ depending, respectively, on whether a_i is even or odd. Figure 4.1**b** illustrates the chain code of the simple object shown in Figure 4.1**a**. One can notice that per definition the chain coded contour lies by half voxel inside the object introducing a systematic error of 0.5 *voxel edge* into the description. To generate it the definition of "object" and "background" in the image is required. *Crack* code has been introduced to overcome these limitations. It is defined to lye exactly on the border of background and object (Figure 4.1, image **c**) and can be simplified by replacing two segments building a corner by a diagonal one (image **d**). While the simplification considerably shortens V (see Table 4.1), it also introduces error into the representation. The choice of a coding technique depend on the application, chain codes are most appropriate in case of area-quantized images. In our application to describe hand segmented organs, we have used unsimplified crack codes since these are closest to the original contour.

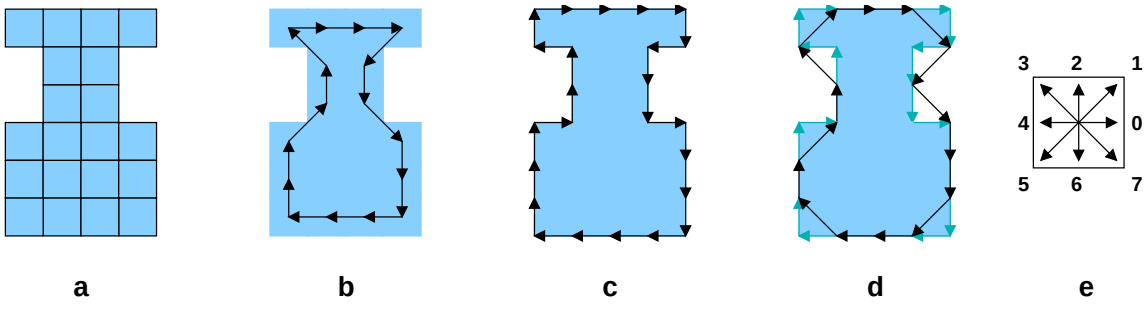

Figure 4.1: Different coding techniques of the simple object shown in **a**. The Freeman code (**b**) is defined to connect the middle points of the border pixels of the object. Images **c** and **d** illustrate the crack code and its simplification, respectively, while **e** shows the assignment of code numbers to directions.

$$\begin{aligned} V_{\text{chain}} &= 0005676644422123 \\ V_{\text{crack}} &= 000064660666444422202242 \\ V_{\text{simplecrack}} &= 0075766544321231 \end{aligned}$$

Table 4.1: Code sequences corresponding to Figures 4.1(**b**,**c**, and **d**,)

4.1.2 Computing Fourier descriptors of a closed polygonal curve

The contour of a simply connected 2-D region is a closed curve which can be mapped onto the parameter space defined by the unit circle Ω_2. The contour is then parametrized by the polar coordinate ϕ, such that the position $\boldsymbol{v}(u)$ on the contour becomes a periodic function of ϕ. This suggests the harmonic Fourier basis being the preferred set of basis functions for the object description. Curves are usually parametrized with their arclength s measured from an arbitrarily chosen starting point on the contour. Thus, when the whole curve has the length L, the point of arclength s from the starting point is assigned to the parameter $\phi = \frac{2\pi s}{L}$. The mathematical treatment is simpler in the complex plane, i.e. when the parameter space is the circle $U = \{u \in \mathbb{C} \mid u^* u = 1\}$ and the mapping $z : U \mapsto \mathbb{C}$ represents the curve. (A full deduction of the complex formulas presented here can be found in Christian Brechbühler's Phd. thesis [Brechbühler-Miškuv 1995].)

The relation between s, ϕ, and u is

$$u = u_0 + j u_1 = e^{j\phi} = e^{\frac{2\pi j s}{L}} \tag{4.3}$$

Fourier expansion

In the complex notation the vector function given in equation 4.1 transforms to the complex valued function $z(u)$. $z(u)$ is represented as a series of complex exponentials.

$$z(u) = \sum_{n=-\inf}^{\inf} z_n u^n \tag{4.4}$$

where the complex coefficient z_n can be expressed in polar notation, i.e.

$$z_n = r_n e^{j\psi_n} \ , \tag{4.5}$$

with $r_n \in \mathbb{R}, r_n \geq 0$, and $\psi \in \mathbb{R}$.

Determining the coefficients

The calculation of z_n for a given contour $z(u)$ is of practical interest. This is given by the formula:

$$z_n = \frac{1}{2\pi} \oint z(u)^{-n} |du| \tag{4.6}$$

In most applications, $z(u)$ describes a polygon and often we are not interested in the center of gravity of the contour. In this case, it is simplest to start from the derivative $\frac{d}{|du|}$ of 4.4 and derive another formula for z_n:

$$z_n = \frac{1}{jn} \oint z'(u)^{-n} |du| \tag{4.7}$$

Computing the Fourier descriptor for a closed polygon

When the curve is given as a polygon (e.g. by its Freeman-code), it can be expressed as a sum of straight line pieces. In the same way the integral 4.7 brakes up into a sum of partial integrals. The M sample points $z(u_k), k = 0 \dots M$ of the Freeman curve define the transitions between the partial integrals. The arclength parametrization of the curve implies that the point $z(u)$ traverses the curve with constant speed; on each individual straight line piece this leads to

$$z'(u) = \frac{\Delta z}{|\Delta u|} = \frac{L \Delta z}{2\pi |\Delta u|} = \text{const} \tag{4.8}$$

This relation significantly simplifies the expression for z_n since the partial integrals of straight line segments can be substituted for their solution.

$$z_n = -\frac{1}{n^2} \sum_{k=0}^{M-1} z'[k] u^{-n} \; |_{u_k}^{u_{k+1}} \tag{4.9}$$

To evaluate this sum, it is enough to calculate one complex exponential for each term. As the lower bound of each term is equal to the upper bound of the previous term, the value of u^{-n} can be reused.

In the case of Freeman-code, Δz can only take eight different values, namely 1, $1 + j$, j, $-1 + j$, -1, $-1 - j$, $-j$, and $1 - j$. The same holds for z': these as well are completely determined by the code $0 \dots 7$.

4.1.3 Normalization in object and parameter space

The normalization proposed by [Kuhl and Giardina 1982] is based on the ellipse defined by the 1st order Fourier descriptors and is carried out both in object and in parameter space. Normalization in object space effects the curve's position, orientation and size, while that in parameter space applies to the curve parametrization behind it. After normalization in object space the center of the 1st order ellipse of a normalized contour concurs with the coordinate origin, its main axis overlaps with the x-axis of the coordinate system has the length of 1. In parameter space the starting point of the parametrization is moved to a standard position defined by the crossing of the 1st order ellipse and its main axis.

The authors derive an error bound on the Fourier approximation which can be used to determine the number of harmonics required by a desired accuracy. Furthermore, they describe a classification and recognition procedure that is applicable to classes of objects which may occur in different orientation, sizes and translation. The following section briefly summarizes the mathematics of Fourier descriptors.

Dependence on starting point

To make the descriptors independent on the starting point of the parametrization, this can be shifted to a standard position, e.g. to the tip of the ellipse defined by the 1st order Fourier descriptors. This can be thought of as a rotation in parameter space U given by the unit circle. The transformation is defined by

$$z_n \,|^V = z_n e^{jn\theta} \,, \tag{4.10}$$

where the notation $|^V$ marks the coefficients resulting from shifting by angle θ.

Dependence on rotational position

In the complex notation, rotation in object space by angle ψ is simply a multiplication by $e^- j\psi$. Applying it to 4.4 immediately reveals the coefficients of the rotated object.

$$z_n \,|^R = z_n e^{j\psi} \tag{4.11}$$

To achieve a standardized position of the curve, it is rotated in a way that its first ellipse's main axis matches the horizontal (real) coordinate axis.

Scale dependence

Scaling the objects by factor α leads to multiplying its coefficients by the same factor:

$$z_n \,|^S = \alpha z_n \tag{4.12}$$

The scaling factor α is usually set to normalize the half major axis to unity, meaning

$$\alpha = \frac{1}{|z_1| + |z_{-1}|} = \frac{1}{r_1 + r_{-1}} \tag{4.13}$$

Invariant Fourier descriptors

Ignoring z_0, that is setting $z_0 \,|^T = 0$, achieves translation invariance. Summing up all standardizations; the invariant coefficients are denoted $\tilde{z}_n$:

$$z_n \,|^{V,R,S,T} = \tilde{z}_n = z_n \frac{e^{j(n\theta - \psi)}}{r_1 + r_{-1}} \tag{4.14}$$

$$\tilde{z}_0 = 0 \tag{4.15}$$

4.1.4 Relations to real valued notation

In the complex notation of Fourier coefficients real and imaginary parts of z_n correspond to the x and y coordinates

$$\begin{pmatrix} x \\ y \end{pmatrix}_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \begin{pmatrix} \sin \frac{2\pi ns}{L} \\ \cos \frac{2\pi ns}{L} \end{pmatrix} \,, \tag{4.16}$$

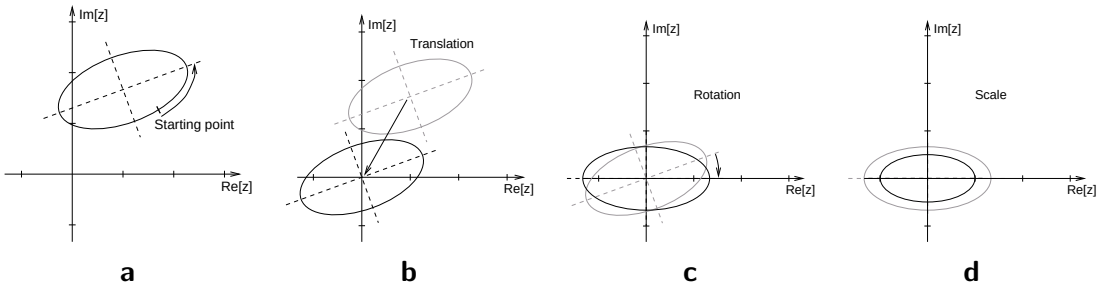

Figure 4.2: Normalization steps of Fourier coefficients; shifting of the starting point to the tip of the ellipse (**a**), moving the center of gravity to the coordinate origin (**b**), rotating the main axis of the ellipse to the real axis (**c**), and finally scaling the half major axis to unity (**d**).

where the real valued coefficients a_n, b_n, c_n, and d_n are defined as follows.

$$
\begin{aligned}
a_n &= \mathrm{Re}z_n + \mathrm{Re}z_{-n} && (4.17)\\
b_n &= -\mathrm{Im}z_n + \mathrm{Im}z_{-n} && (4.18)\\
c_n &= \mathrm{Im}z_n + \mathrm{Im}z_{-n} && (4.19)\\
d_n &= \mathrm{Re}z_n - \mathrm{Re}z_{-n} && (4.20)\\
& && (4.21)
\end{aligned}
$$

4.2 Description of surfaces by spherical harmonic functions

The problem of finding a similarly homogeneous parametrization of arbitrarily shaped surfaces proved to be more difficult. [Brechbühler *et al.* 1995] introduced only recently a new surface parametrization technique which can be considered as a complete generalization of Kuhl and Giardina's technique to three dimensions.

Earlier methods for mapping an object surface onto a sphere have been limited to represent only star-shaped or convex objects, as they start from an initial radial surface function $r(\theta, \phi)$. [Staib and Duncan 1992b] discuss the use of a parameter space with torus topology, which can be deformed into a tube by squeezing the torus cross-section to a thin ribbon. Closed surfaces are obtained by considering tubes whose ends close up to a point. This approach illustrates some principal difficulties which can also be found in other parametrization techniques.

- Warping a torus to a closed surface poses the problem that the *parameters have different rules*. One parameter defines a kind of spine along which cross-sections are stacked up.
- Squeezing a circle to line results in a *nonhomogeneous distribution of parameters* on the object surface.
- Warping a torus to a tube and finally to a closed surface causes the parametrization does *not* result in a *one-to-one mapping* of surface points to parameters.