Lecture:
Shape Analysis
Elliptic (Fourier) Harmonics

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Materials

- Paper Kuhl & Giardina CGIP 1982
- Notes Andras Kelemen on elliptic harmonics
- Paper Staib&Duncan, PAMI 1992
- Hand-written notes G. Gerig

Boundary parametrization

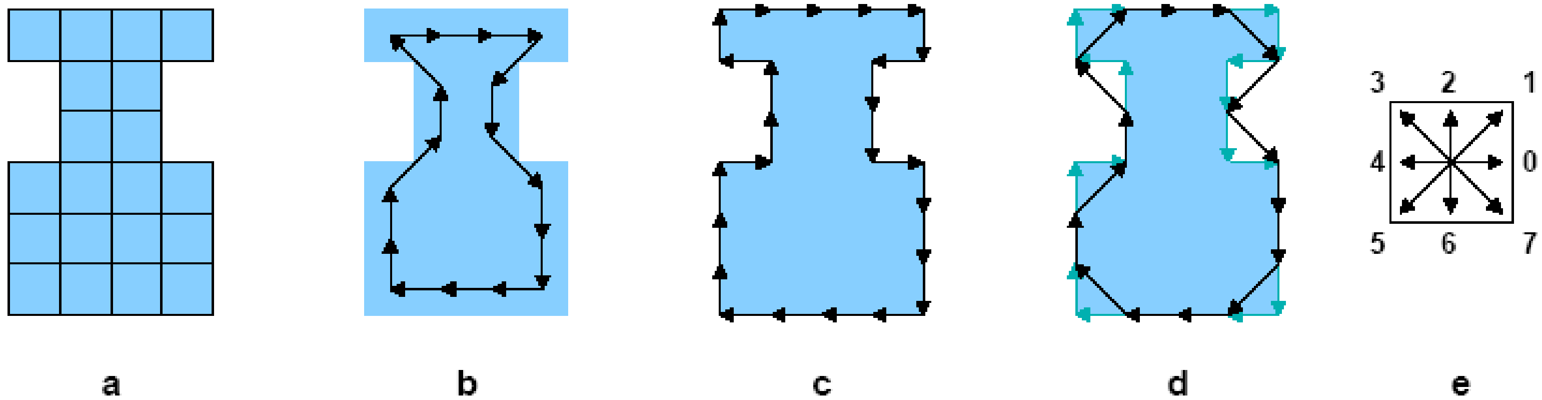

Figure 4.1: Different coding techniques of the simple object shown in **a**. The Freeman code (**b**) is defined to connect the middle points of the border pixels of the object. Images **c** and **d** illustrate the crack code and its simplification, respectively, while **e** shows the assignment of code numbers to directions.

$$
\begin{aligned}
V_{\text{chain}} &= 00056766444222123 \\
V_{\text{crack}} &= 0000646606666444422202242 \\
V_{\text{simplecrack}} &= 0075766544321231
\end{aligned}
$$

Table 4.1: Code sequences corresponding to Figures 4.1(**b**,**c**, and **d**,)

Boundary Parametrization

$$v(s) = \begin{pmatrix} v_1(s) \\ v_2(s) \end{pmatrix} = \begin{pmatrix} x(s) \\ y(s) \end{pmatrix} \qquad u = u_0 + ju_1 = e^{j\phi} = e^{\frac{2\pi j s}{L}}$$

Fourier expansion

In the complex notation the vector function given in equation 4.1 transforms to the complex valued function $z(u)$. $z(u)$ is represented as a series of complex exponentials.

$$z(u) = \sum_{n=-\inf}^{\inf} z_n u^n \tag{4.4}$$

where the complex coefficient z_n can be expressed in polar notation, i.e.

$$z_n = r_n e^{j\psi_n} \ , \tag{4.5}$$

with $r_n \in \mathbb{R}, r_n \geq 0$, and $\psi \in \mathbb{R}$.

Determining Coefficients

Determining the coefficients

The calculation of z_n for a given contour $z(u)$ is of practical interest. This is given by the formula:

$$z_n = \frac{1}{2\pi} \oint z(u)^{-n} |du| \tag{4.6}$$

In most applications, $z(u)$ describes a polygon and often we are not interested in the center of gravity of the contour. In this case, it is simplest to start from the derivative $\frac{d}{|du|}$ of 4.4 and derive another formula for z_n:

$$z_n = \frac{1}{jn} \oint z'(u)^{-n} |du| \tag{4.7}$$

$$z_n = -\frac{1}{n^2} \sum_{k=0}^{M-1} z'[k] u^{-n} \Big|_{u_k}^{u_{k+1}} \tag{4.9}$$

Harmonic Contributions

Harmonic contributions to the contour In equation (2.9) we match terms with opposite indices in pairs

$$z(u) \quad = \quad z_0 + \sum_{n=1}^{\infty} \underbrace{z_n u^n + z_{-n} u^{-n}}_{elli_n(u)} \tag{2.17}$$

All harmonic contributions – except z_0 – describe an ellipse when taken by themselves. The ellipse $elli_n(u)$ is covered n times when u runs over the circle U. The vertex of the ellipse, i.e. the maximum of the absolute value of $elli_n$, is reached where both terms of the sum have the same phase; at this point the triangle inequality

$$|elli_n(u)| \quad \leq \quad |z_n u^n| + \left|z_{-n} u^{-n}\right| \tag{2.18}$$

holds with equality.

Normalization

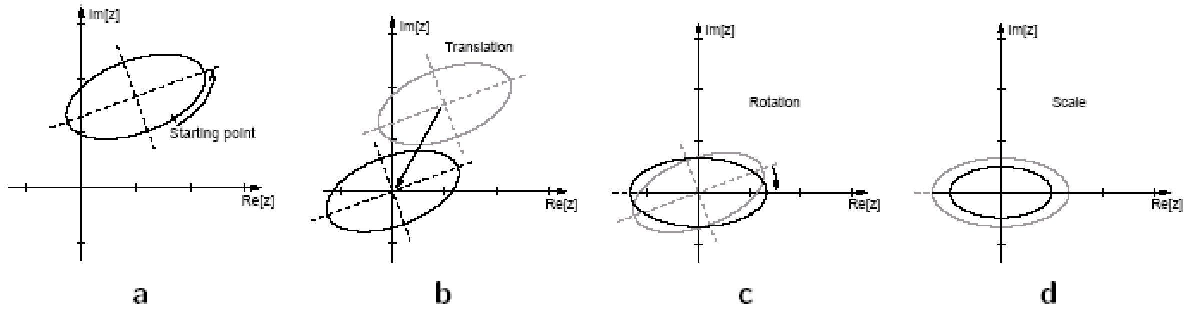

Figure 4.2: Normalization steps of Fourier coefficients; shifting of the starting point to the tip of the ellipse (**a**), moving the center of gravity to the coordinate origin (**b**), rotating the main axis of the ellipse to the real axis (**c**), and finally scaling the half major axis to unity (**d**).

Translation		2	$\tilde{z}_0 = 0$
Rotation	ψ	1	$\tilde{z}_1, \tilde{z}_{-1} \in \mathbb{R}$
Starting point	$\lambda = \frac{L\theta}{2\pi}$	1	
Scale		1	$\tilde{z}_1 + \tilde{z}_{-1} = 1$

Invariant Fourier descriptors

Ignoring z_0, that is setting $z_0 \mid^T = 0$, achieves translation invariance. Summing up all standardizations; the invariant coefficients are denoted $\tilde{z}_n$:

$$z_n \mid^{V,R,S,T} = \tilde{z}_n = z_n \frac{e^{j(n\theta - \psi)}}{r_1 + r_{-1}} \quad (4.14)$$

$$\tilde{z}_0 = 0 \quad (4.15)$$

Truncation / Classification

Truncating the expansion If only for practical reasons, the infinite sum (2.9) must be truncated at some maximum degree, say N. We define the partial sum $\check{z}$ and the complex-valued deviation $f(u)$ as follows.

$$\check{z}(u) = \sum_{n=-N}^{N} z_n u^n \tag{2.37}$$

$$f(u) = z(u) - \check{z}(u) = \sum_{|n|>N} z_n u^n \tag{2.38}$$

Classification of objects

Model based object recognition is an important application of shape descriptors. For each model object g, a descriptor, i.e. a collection $\{\tilde{z}_n\}[g]$, is computed in the so-called training phase. In the same way a descriptor $\{\hat{z}_n\}$ is determined for the unknown object. We define a *classification distance* (or its square, respectively).

$$D[g] = \sum_{n=-N}^{N} |\tilde{z}_n[g] - \hat{z}_n|^2 \tag{2.42}$$

We decide for the model g with minimal $D[g]$.

Reconstruction from Elliptic Descriptors