Lecture:
Shape Analysis
Moment Invariants

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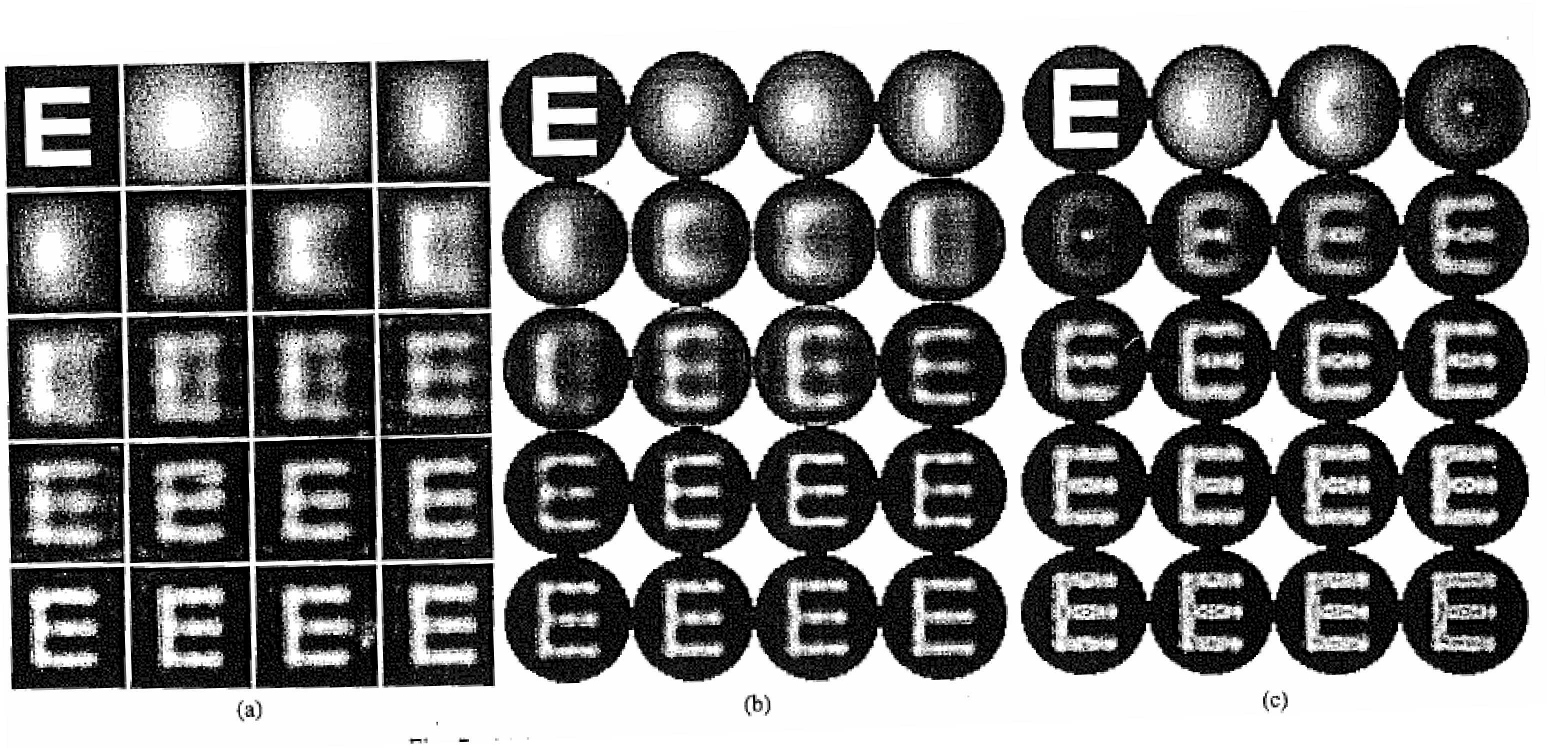
CS 7960, Spring 2010

References

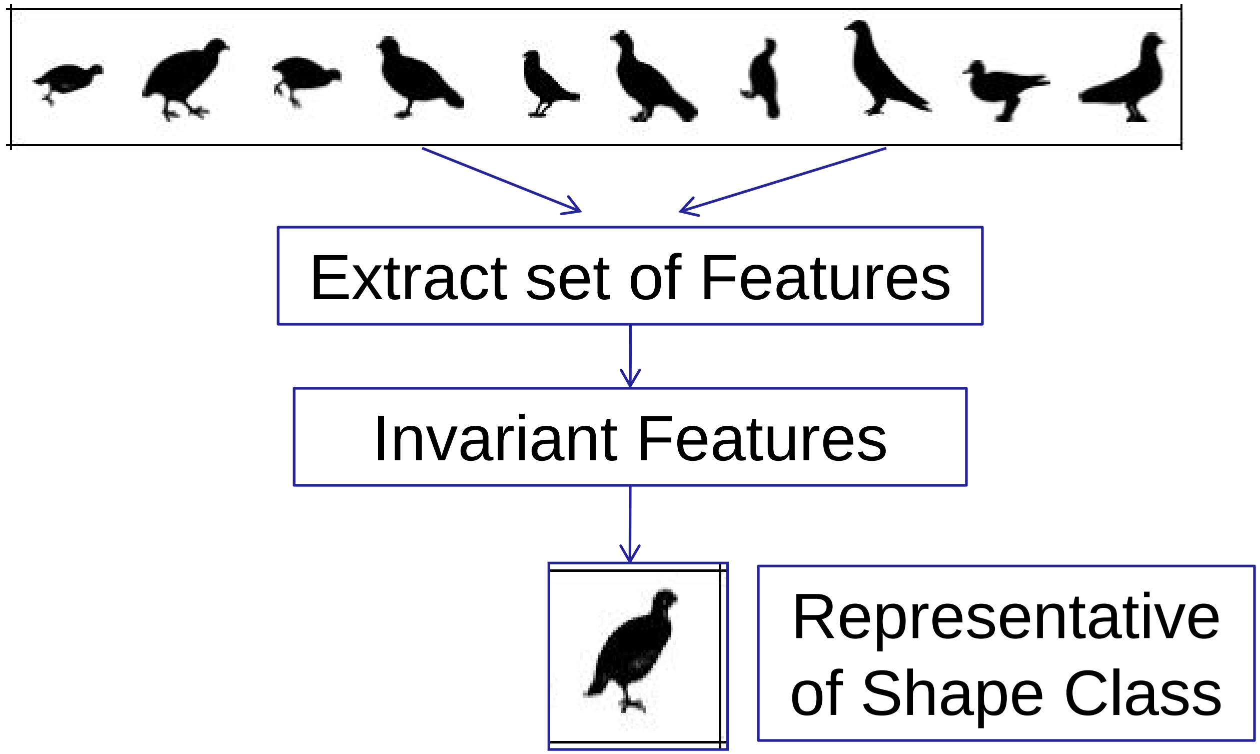
- Cho-Hua Teh, Roland T. Chin, On Image Analysis by the Methods of Moments, IEEE T-PAMI, 1988
- Ming-Kuei Hu, Visual Pattern Recognition by Moment Invariants, IEEE Transactions on Information Theory, 1962
- M.R. Teague, Image analysis via the general theory of moments, J. Opt. Soc. Am. Vol. 70, No. 8, Aug 1980, pp. 920ff
- Materials Erik W. Anderson, SCI PhD student

Motivation

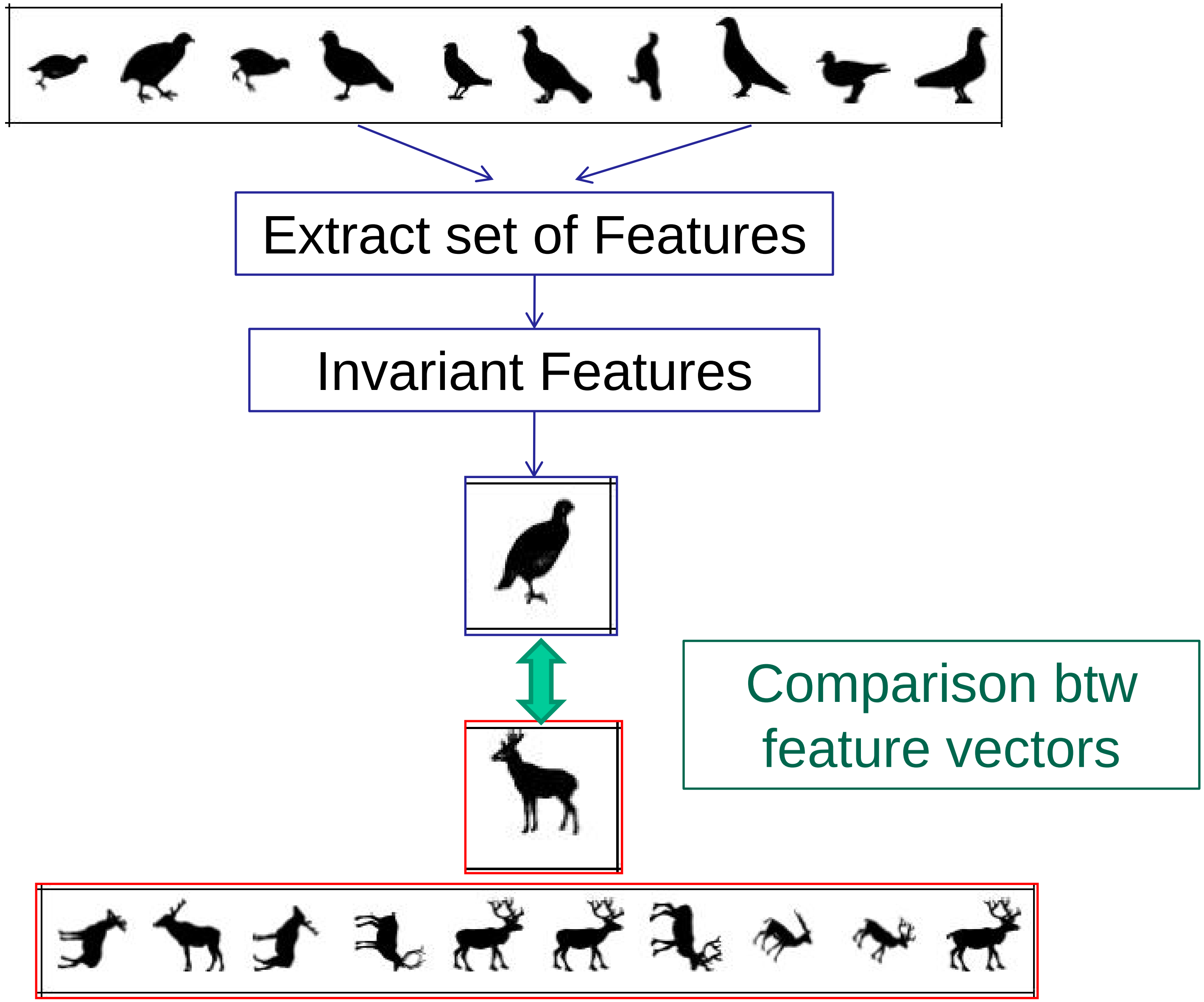


Reconstruction of letter E by a) Legendre Moments, b) Zernike Moments, and c) pseudo Zernike Moments (from Teh/Chin 1988)

Basic Concept



Basic Concept ctd.



Basic Concept ctd.

Classify (recognize) each shape into one of the shape classes

Method

- Moments m_{pq}: projection of image ϱ(x,y) to basis $x^p y^q$.
- ϱ(x,y): piecewise continuous function with non-zero values in a portion of the plane = image.

$$m_{pq} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^p y^q \rho(x,y)\,dx\,dy$$

- Raw image moments:

$$m_{pq} = \sum_x \sum_y x^p y^q f(x,y)$$

$f(x,y)$

Raw Moments

$$m_{pq} = \sum_x \sum_y x^p y^q f(x,y)$$

- M_{00}: ??
- M_{10}: ??
- M_{01}: ??

- Centroid coordinates: ??

Raw Moments

$$\boxed{m_{pq} = \sum_x \sum_y x^p y^q f(x,y)}$$

- M_{00}: area/volume, #pixels if binary image
- M_{10}: sum over x
- M_{01}: sum over y

- Centroid coordinates:

$$\bar{x} = \frac{M_{10}}{M_{00}} \qquad \bar{y} = \frac{M_{01}}{M_{00}}$$

Translation Invariance

- **Statistics:** nth moment about the mean, or nth central moment of a random variable X is defined as:

$$\mu_n = E[(X - E[X])^n] = \int_{-\infty}^{\infty} (x - \mu)^n f(x)\,dx$$

Translation Invariance

- **Statistics:** n^{th} moment about the mean, or n^{th} central moment of a random variable X is defined as:

$$\mu_n = E[(X - E[X])^n] = \int_{-\infty}^{\infty} (x-\mu)^n f(x)\,dx$$

- Extension to 2D, discrete sampling:

$$\boxed{\mu_{pq}} = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} (x-\bar{x})^p (y-\bar{y})^q f(x,y)$$

$$\bar{x} = \frac{M_{10}}{M_{00}} \qquad \bar{y} = \frac{M_{01}}{M_{00}}$$

Central Moments

$$\underline{\underline{\mu_{pq}}} = \iint (x-\bar{x})^p (y-\bar{y})^q f(x,y)\,dx\,dy$$

$$\overset{*}{=} \iint \sum_{r=0}^{p} \binom{p}{r} x^r (-\bar{x})^{(p-r)} \sum_{s=0}^{q} \binom{q}{s} -\bar{y}^{(q-s)} f(x,y)\,dx\,dy$$

$$= \sum_{r=0}^{p} \sum_{s=0}^{q} \binom{p}{r}\binom{q}{s} (-\bar{x})^{p-r} (-\bar{y})^{q-s} \cdot \underbrace{M_{rs}}_{\text{absolute Momente}}$$

$$* \quad (a+b)^k = \sum_{r=0}^{k} \binom{k}{r} a^r \cdot b^{(k-r)} \qquad \binom{k}{r} = \frac{k!}{r!\,(k-r)!}$$

Central Moments ctd.

$$\mu_{00} = M_{00}$$

$$\mu_{10} = \iint (x - \bar{x})\, f(x,y)\, dx\, dy = \underbrace{\iint x \cdot f(x,y)\, dx\, dy}_{M_{10}} \underbrace{- \bar{x} \int f(x,y)\, dx\, dy}_{-\frac{M_{10}}{M_{00}} \cdot M_{00}} = \emptyset$$

$$\mu_{01} = \emptyset$$

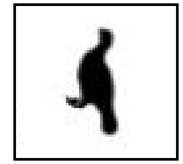
$$\mu_{20} = M_{20} - \bar{x}\, M_{10}$$

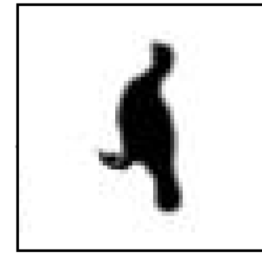
$$\vdots$$

→ central moments constructed from raw moments

Scale Invariance

- f'(x,y): new image scaled by λ





$$\Rightarrow \quad f'(x,y) = f\left(\frac{x}{\lambda}, \frac{y}{\lambda}\right)$$

Variablentransformation:

$$x' = \frac{x}{\lambda} \qquad y' = \frac{y}{\lambda} \qquad dx = \lambda\, dx' \quad dy = \lambda\, dy'$$

$$\boxed{\underline{\underline{\mu'_{pq}}}} = \iint x^p y^q \cdot f\left(\frac{x}{\lambda}, \frac{y}{\lambda}\right) dx\, dy$$

$$= \iint (\lambda x')^p (\lambda y')^q f(x', y')\, \lambda^2\, dx'\, dy'$$

$$= \lambda^p \lambda^q \lambda^2 \iint x'^p\, y'^q\, f(x', y')\, dx'\, dy'$$

$$= \boxed{\lambda^{(p+q+2)} \cdot \mu_{pq}}$$

Scale Invariance ctd.

- Concept: Set total area to 1

$$\mu'_{00} = \lambda^2 \mu_{00} \doteq 1$$

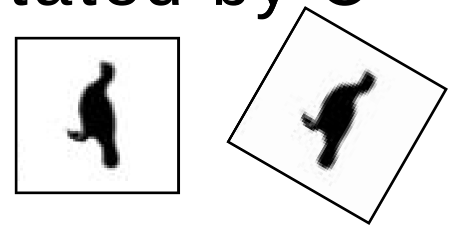
$$\Rightarrow \lambda = \frac{1}{\sqrt{\mu_{00}}} = \mu_{00}^{-\frac{1}{2}}$$

- Scaling invariant modes:

$$\underline{\underline{\eta_{pq}}} = \frac{1}{\mu_{00}^{\left(\frac{p+q+2}{2}\right)}} \cdot \mu_{pq}$$

Rotation Invariance

- f'(x,y): new image rotated by Θ



$$f'(x,y) = f(x\cos\Theta + y\sin\Theta, -x\sin\Theta + y\cos\Theta)$$

Variablentransformation:

$$x' = x\cos\Theta + y\sin\Theta$$
$$y' = -x\sin\Theta + y\cos\Theta$$

$$\underline{x}' = R \cdot \underline{x}$$

$$\underline{x} = R^{-1} \cdot \underline{x}'$$

$$x = x'\cos\Theta - y'\sin\Theta$$
$$y = x'\sin\Theta + y'\cos\Theta$$

$$dx = \cos\Theta \, dx'$$
$$dy = \cos\Theta \, dy'$$

Rotation Invariance ctd.

$$\underline{\mu'_{pq}} = \iint x^p y^q f'(x, y)\, dx\, dy$$

$$= \iint x^p y^q f(x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta)\, dx\, dy$$

$$= \iint (x'\cos\theta - y'\sin\theta)^p (x'\sin\theta + y'\cos\theta)^q f(x', y') \cos^2\theta\, dx'\, dy'$$

$$= \left(\text{see Teague p. 925} \right) = \sum_{r=0}^{p} \sum_{s=0}^{q} (-1)^{q-s} \binom{p}{r} \binom{q}{s} \cos\theta^{(p-r+s)} \sin\theta^{(q+r-s)} \cdot \mu_{p+q-r-s, r+s}$$

$$\theta = \frac{1}{2} \arctan\left(\frac{2\mu_{11}}{\mu_{20} - \mu_{02}} \right)$$

Rotation Invariance ctd.

$$\underline{\underline{\mu'_{pq}}} = \iint x^p y^q f'(x,y)\, dx\, dy$$

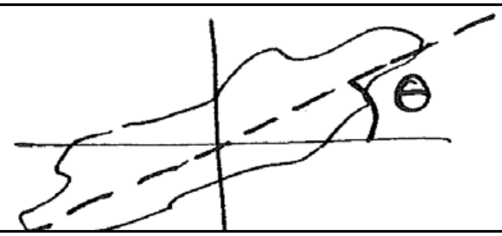
$$= \iint x^p y^q f(x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta)\, dx\, dy$$

$$= \iint (x'\cos\theta - y'\sin\theta)^p (x'\sin\theta + y'\cos\theta)^q f(x', y') \cos^2\theta\, dx'\, dy'$$

$= \Big($ see Teague p. 925 $\Big)$ $\displaystyle = \sum_{r=0}^{p}\sum_{s=0}^{q} (-1)^{q-s} \binom{p}{r}\binom{q}{s} \cos\theta^{(p-r+s)} \sin\theta^{(q+r-s)} \cdot \mu_{p+q-r-s,\, r+s}$

- Rotation to first axis of inertia:

$$\theta = \frac{1}{2}\arctan\left(\frac{2\mu_{11}}{\mu_{20}-\mu_{02}}\right)$$



$$\Rightarrow \mu_{11}^{\mathrm{rot}\,\theta} \overset{!}{=} 0$$

Rotation Invariance ctd.

$$\begin{pmatrix} \mu'_{20} & \mu'_{11} \\ \mu'_{11} & \mu'_{02} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \mu_{20} & \mu_{11} \\ \mu_{11} & \mu_{02} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

- Discussion Rotation Invariance:
  - Basis $\{x^p y^q\}$ doesn't have simple rotation properties
  - Building of moments that are invariant to rotation is very difficult
- Solution: New function system that has better rotational properties

Orthogonal Invariants by Hu method

$$
\begin{aligned}
&\mu_{20} + \mu_{02}, \\
&(\mu_{20} - \mu_{02})^2 + 4\mu_{11}^2, \\
&(\mu_{30} - 3\mu_{12})^2 + (3\mu_{21} - \mu_{03})^2, \\
&(\mu_{30} + \mu_{12})^2 + (\mu_{21} + \mu_{03})^2, \\
&(\mu_{30} - 3\mu_{12})(\mu_{30} + \mu_{12})[(\mu_{30} + \mu_{12})^2 - 3(\mu_{21} + \mu_{03})^2] \qquad (61) \\
&\qquad\qquad + (3\mu_{21} - \mu_{03})(\mu_{21} + \mu_{03}) \\
&\qquad\qquad \cdot [3(\mu_{30} + \mu_{12})^2 - (\mu_{21} + \mu_{03})^2], \\
&(\mu_{20} - \mu_{02})[(\mu_{30} + \mu_{12})^2 - (\mu_{21} + \mu_{03})^2] \\
&\qquad\qquad + 4\mu_{11}(\mu_{30} + \mu_{12})(\mu_{21} + \mu_{03}),
\end{aligned}
$$

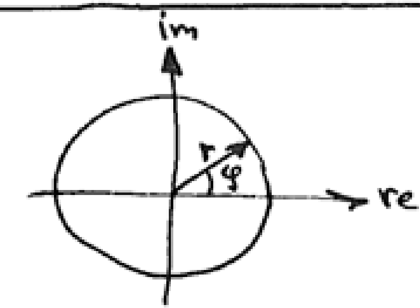
and one skew orthogonal invariants,

$$
\begin{aligned}
&(3\mu_{21} - \mu_{03})(\mu_{30} + \mu_{12})[(\mu_{30} + \mu_{12})^2 - 3(\mu_{21} + \mu_{03})^2] \\
&\quad - (\mu_{30} - 3\mu_{12})(\mu_{21} + \mu_{03})[3(\mu_{30} + \mu_{12})^2 - (\mu_{21} + \mu_{03})^2]. \qquad (62)
\end{aligned}
$$

- Invariants are independent of position, size and orientation
- However: This is not a complete set, and there is no simple way for reconstruction!

Complex Moments

- Abu-Mostafa, Yaser S., and Demetri Psaltis. Image normalization by complex moments; T-PAMIJan 85 46-55



$$z = a + ib = r(\cos\varphi + i\sin\varphi)$$
$$= r \cdot e^{i\varphi}$$
$$z^p = (a+ib)^p = r^p\, e^{ip\varphi}$$

$$C_{pq} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underbrace{\underbrace{(x+iy)^p}_{r^p \cdot e^{ip\varphi}} \underbrace{(x-iy)^q}_{r^q \cdot e^{-iq\varphi}}}_{r^{(p+q)} \cdot e^{i(p-q)\varphi}} f(x,y)\, dx\, dy$$

Complex Moments ctd.

$$C_{pq} = \int_0^{2\pi}\int_0^{\infty} \underbrace{r^{(p+q)} e^{i(p-q)\varphi}}_{\text{"CM-kernels"}} f(r\cos\varphi, r\sin\varphi)\, r\,dr\,d\varphi$$

Notation: p+q=n: Order

p-q=l: Repetition

$$C_{pq} \longrightarrow C_n^{\ell} \qquad\qquad C_{qp} \longrightarrow C_n^{-\ell}$$

Relationship to Raw Moments

$$C_3^1 = C_{21}: \quad (x+iy)^2(x-iy)$$

$$= \underbrace{(x^3 + xy^2)}_{Re(C_{21})} + i\underbrace{(x^2y + y^3)}_{Im(C_{21})} = a + ib$$

$$Re(C_{21}) = M_{30} + M_{12}$$

$$Im(C_{21}) = M_{21} + M_{03}$$

Properties of CM

$$C_{qp} = C_n^{-\ell} = \int_0^{2\pi}\int_0^{\infty} r^n e^{-i\ell\varphi} f(r,\varphi)\, r\,dr\,d\varphi$$

$$\Rightarrow \quad \left.\begin{aligned} C_n^{-\ell} &= C_n^{\ell\,*} \\ C_{qp} &= C_{pq}^{*} \end{aligned}\right\} \text{conjugate complex}$$

Translation Invariance

$$C_1^1 = C_1^{-1} \overset{!}{=} 0$$

$$C_1^1 = \iint r \cdot e^{i\varphi} f(r,\varphi)\, r\, dr\, d\varphi = \iint (x+iy)\, f(x,y)\, dx\, dy$$

$$\left.\begin{aligned} \mathrm{Re}(C_1^1) &= M_{10} \\ \mathrm{Im}(C_1^1) &= M_{01} \end{aligned}\right\} \Rightarrow M_{10} = M_{01} = \emptyset$$

Setting M10 and M01 to 0 makes series translational invariant

Scale Invariance

$$C_0^0 = \iint f(x,y)\,dx\,dy \overset{!}{=} 1$$

(see earlier discussion with raw moments)

CM under Rotation

$$C_n^l\big|^{rot} = \iint r^n e^{il\varphi} f'(r,\varphi)\, r\,dr\,d\varphi$$

Rotation φ_0 : $f'(r,\varphi) = f(r, \varphi + \varphi_0)$

$\varphi' = \varphi + \varphi_0 \qquad d\varphi' = d\varphi$

$r' = r \qquad dr' = dr$

$$\Rightarrow C_n^l\big|^{rot} = \iint r^n e^{il\varphi} f(r, \varphi + \varphi_0)\, r\,dr\,d\varphi$$

$$= \iint r'^n e^{il(\varphi' - \varphi_0)} f(r', \varphi')\, r'\,dr'\,d\varphi'$$

$$= e^{-il\varphi_0} \cdot \underbrace{\iint r'^n \cdot e^{il\varphi'} f(r', \varphi')\, r'\,dr'\,d\varphi'}_{C_n^l}$$

$$\Rightarrow \boxed{C_n^l\big|^{rot} = e^{-il\varphi_0} \cdot C_n^l}$$

CMs have very clear, simple rotational properties

Set of CM's

Order n	C_{pq}	C_n^{ℓ}
0	C_{00}	C_0^0
1	C_{10} C_{01}	C_1^1 C_1^{-1}
2	C_{20} C_{11} C_{02}	C_2^2 C_2^0 C_2^{-2}
3	C_{30} C_{21} C_{12} C_{03}	C_3^3 C_3^1 C_3^{-1} C_3^{-3}
4	C_{40} C_{31} C_{22} C_{13} C_{04}	C_4^4 C_4^2 C_4^0 C_4^{-2} C_4^{-4}
5	. . . . .	

#coefficients order n: n+1 CM's

#coefficients till order n: $\sum_{k=0}^{n}(k+1) = \frac{(n+1)(n+2)}{2}$

CMs with Rotation Invariance

- Building of algebraic combination of CMs, so that rotational component disappears

$$C_h^{\ell}\big|^{rot} \cdot \left(C_{h'}^{\ell'}\big|^{rot}\right)^k = C_h^{\ell} \cdot \left(C_{h'}^{\ell'}\right)^k \cdot e^{-i\ell\varphi_0} \cdot e^{-i\ell'\varphi_0 \cdot k}$$
$$= C_h^{\ell} \cdot \left(C_{h'}^{\ell'}\right)^k \cdot e^{-i(\ell+\ell'\cdot k)\varphi_0}$$

(Rotation Invariants: $\ell + \ell' \cdot k = \emptyset$)

$$\boxed{\text{Rotation Invariants:} \quad C_h^{\ell} \cdot \left(C_{h'}^{\ell'}\right)^k + C_h^{-\ell} \cdot \left(C_{h'}^{-\ell'}\right)^k \quad \text{für} \quad \ell + \ell' \cdot k = \emptyset}$$

CMs with Rotation Invariance

Rotation Invariants: $C_n^{\ell}\cdot\left(C_{n'}^{\ell'}\right)^k + C_n^{-\ell}\cdot\left(C_{n'}^{-\ell'}\right)^k$ für $\ell + \ell' k = 0$

$k=0$: $\ell = 0 \Rightarrow C_n^0 + C_n^0$ Bsp: C_0^0 ; C_4^0

$k=1$: $\ell = -\ell' \Rightarrow C_n^{\ell}\cdot C_{n'}^{-\ell} + C_n^{-\ell}\cdot C_{n'}^{\ell}$

a) $n=n' \Rightarrow 2C_n^{\ell}\cdot C_n^{-\ell} = 2C_n^{\ell}\cdot C_n^{\ell\,*}$ Bsp: $C_3^1\cdot C_3^{-1}$ (Betrag)

b) $n \neq n' \Rightarrow C_n^{\ell}\cdot C_{n'}^{-\ell} + C_n^{-\ell}\cdot C_{n'}^{\ell}$ Bsp: $C_4^2\cdot C_2^{-2} + C_4^{-2}\cdot C_2^2$

$k=2$: $\ell = -2\ell' \Rightarrow C_n^{\ell}\cdot\left(C_{n'}^{-\frac{\ell}{2}}\right)^2 + C_n^{-\ell}\cdot\left(C_{n'}^{\frac{\ell}{2}}\right)^2$

a) $n=n'$ Bsp: $C_4^4\cdot\left(C_4^{-2}\right)^2 + C_4^{-4}\left(C_4^2\right)^2$ (Linearkomb.!)

b) $n \neq n'$ Bsp: $C_4^4\cdot\left(C_2^{-2}\right)^2 + C_4^{-4}\left(C_2^2\right)^2$

$C_2^2\cdot\left(C_3^{-1}\right)^2 + C_2^{-2}\left(C_3^1\right)^2$

$\vdots$

CMs with Rotation Invariance

Order n	
0	$C_0^0 \stackrel{!}{=} 1$
1	$C_1^1 = C_1^{-1} \stackrel{!}{=} \phi$
2	$C_2^2 \cdot C_2^{-2} \quad C_2^0 \stackrel{!}{=} \phi$
3	$[C_3^3 \cdot C_3^{-3}]$ $[C_3^1 \cdot C_3^{-1}]$ $[(C_3^1)^2 \cdot C_2^{-2} + (C_3^{-1})^2 \cdot C_2^2]$ $[C_3^3 \cdot (C_1^{-1})^3 + C_3^{-3} \cdot (C_1^1)^3]$
4	$[C_4^0]$ $[C_4^4 \cdot C_4^{-4}]$ $[C_4^2 \cdot C_4^{-2}]$ $[C_4^2 \cdot (C_3^{-1})^2 + C_4^{-2} (C_3^1)^2]$
5	. . . $[C_4^4 \cdot (C_2^{-2})^2 + C_4^{-4} (C_2^2)^2]$
⋮	

$\Big\}$ Normalization to standard representation (orders 0–2)

Rotation to invariant position

$$C_2^2 = C_{20} = \iint r^2 \cdot e^{i 2 \varphi_2^2} f(r,\varphi)\, r\, dr\, d\varphi$$

$$= \iint (x+iy)^2 f(x,y)\, dx\, dy$$

$$= \iint \left((x^2-y^2) + i\,(2xy) \right) f(x,y)\, dx\, dy$$

$$\mathrm{Re}(C_2^2) = \mu_{20} - \mu_{02}$$

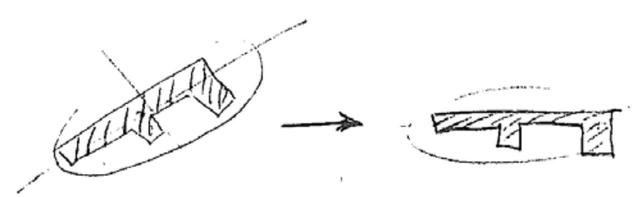
$$\mathrm{Im}(C_2^2) = 2\mu_{11}$$

$$\tan(\varphi_2^2) = \frac{2\mu_{11}}{\mu_{20}-\mu_{02}}$$

$$\varphi_2^2 \overset{!}{=} 0$$

Eliminate rotational part of 2nd order ellipsoid

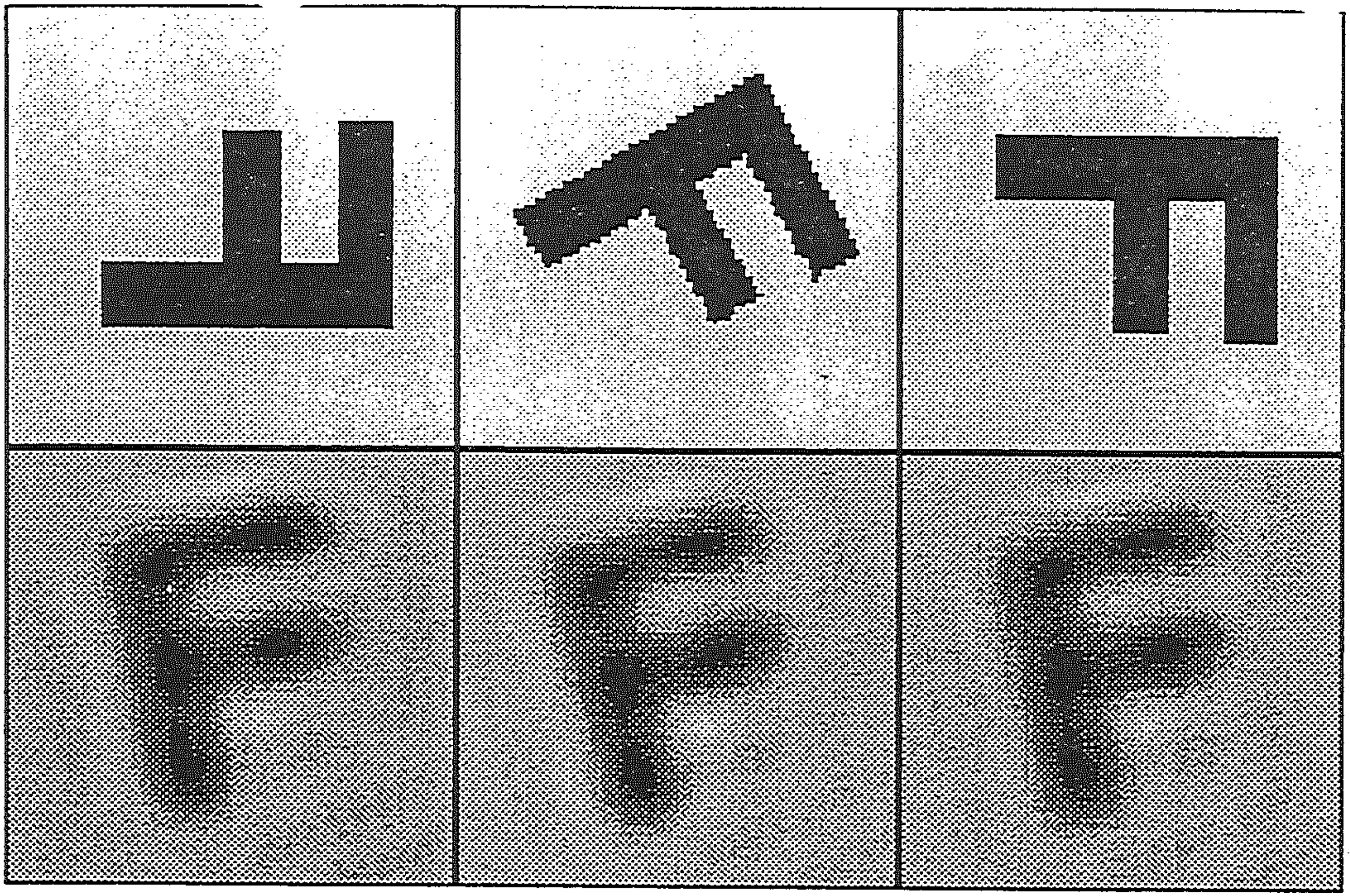
$$\rightarrow \boxed{\begin{array}{l} \mu_{11} \overset{!}{=} 0 \\ \mu_{20} > \mu_{02} \end{array}}$$



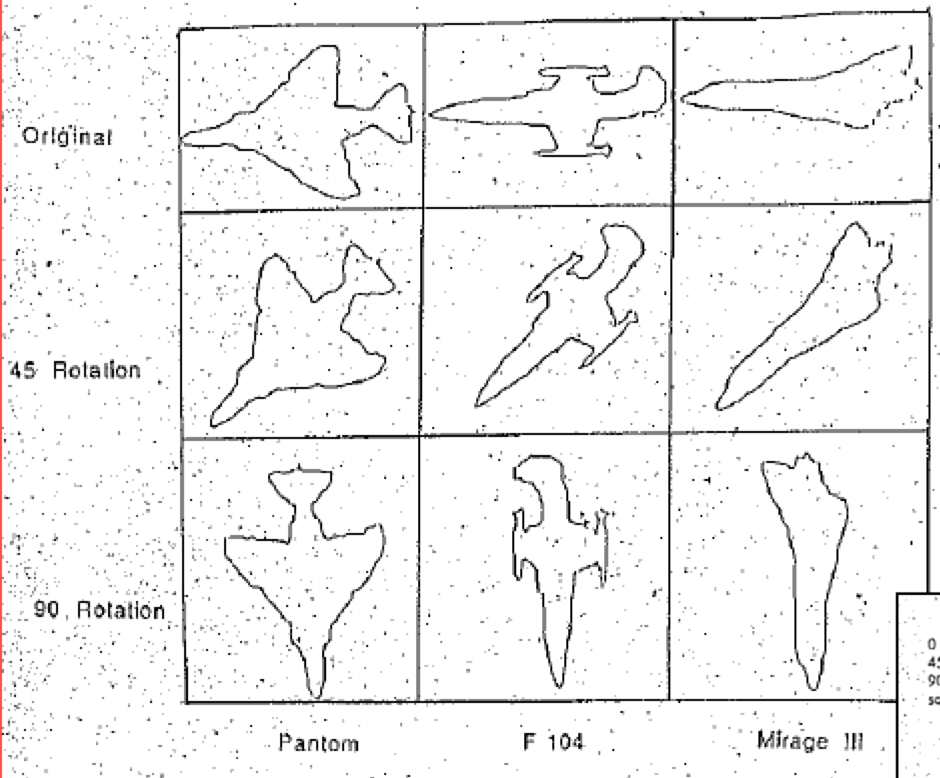
Reconstruction

- Inverse generation of representative shape from normalized moments.
- Building of normal model as shape template for equivalence class.
- Procedure: Systematic reconstruction of phase and coefficients of normalized shape from invariant moments.

Example: Reconstruction from invariant CMs (20th order)



Example: Airplane Recognition



	M1	M2	M3	M4	M5	M6
0 degree	0.4036	0.1140	0.4023	0.1490	0.0361	0.0427
45 degree	0.4849	0.1647	0.4832	0.2108	0.0670	0.0794
90 degree	0.4036	0.1140	0.4023	0.1490	0.3610	0.0427
scale 2X	0.4831	0.1840	2.2655	1.3179	2.4730	0.5160

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	M1	M2	M3	M4	M5	M6
0 degree	0.3390	0.0727	0.0742	0.0111	0.0003	0.0020
45 degree	0.4296	0.1207	0.1920	0.0470	0.0044	0.0160
90 degree	0.3390	0.0727	0.0742	0.0111	0.0003	0.0020
scale 2X	0.3323	0.0660	0.5063	0.1250	0.0310	0.0325

Phantom

	M1	M2	M3	M4	M5	M6
0 degree	0.6973	0.4453	2.5966	1.8089	3.9139	1.1642
45 degree	0.8623	0.6836	3.4007	2.3510	6.6320	1.8720
90 degree	0.6973	0.4453	2.5660	1.8089	3.9139	1.1642
scale 2X	0.6362	0.3661	9.8817	6.9978	58.104	4.0936

Mirage III

Classification

- Image I(x,y) → set of invariants = feature vector **v**
- Statistical pattern recognition: Clustering in multi-dimensional feature space

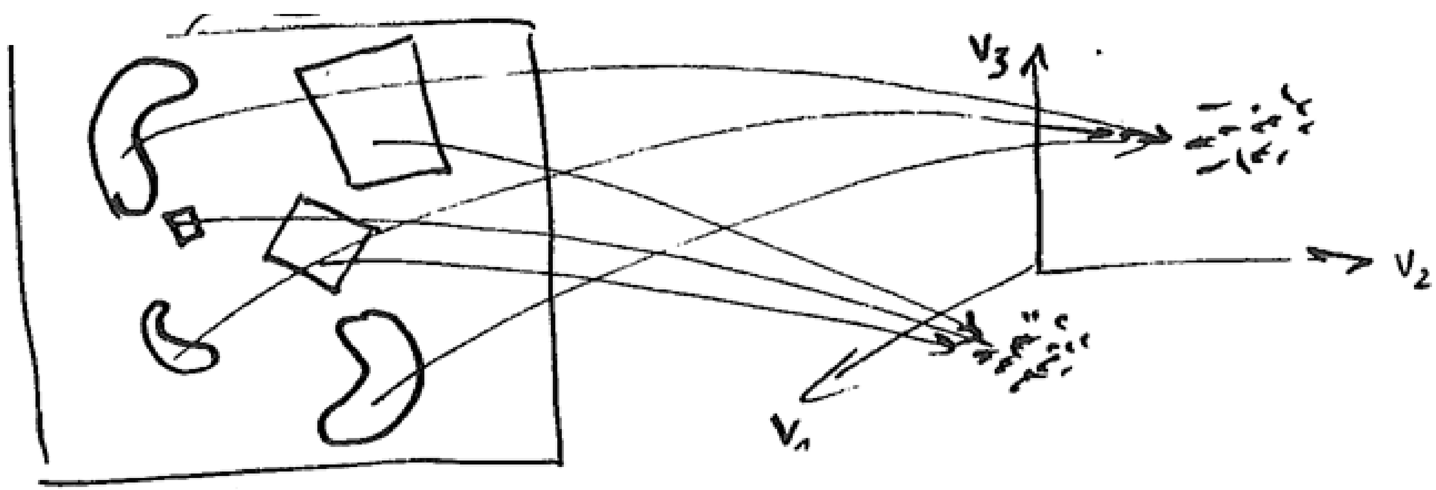

Image space Feature space

- Criteria: good discrimination, small set of features (→ Zernike, pseudo Zernika, Teh/Chin)

Zernike Polynomials

So far: Non-orthogonal basis: Set of moments is complete, but new higher orders influence lower orders.. **Solution:** Orthogonal basis: Zernike Polynomials: Teh & Chin, 1988

Zernicke Polynomials: $V_n^{\ell}(r,\Theta) = R_n^{\ell}(r) \cdot e^{j\ell\Theta}$

Orthogonality: $\underbrace{\int_0^{2\pi}\int_0^{1}}_{\text{Unit disk}} V_n^{\ell *}(r,\Theta) \cdot V_m^{k}(r,\Theta)\, r\,dr\,d\Theta = \frac{\pi}{n+1}\delta_{mn}\delta_{k\ell}$

$$A_n^{\ell} = \frac{n+1}{\pi}\int_0^{2\pi}\int_0^{\infty} R_n^{\ell}(r)\, e^{-j\ell\Theta} f(r,\Theta)\, r\,dr\,d\Theta$$

Same rotational properties as CMs, building of invariants is equivalent

Zernike Polynomials

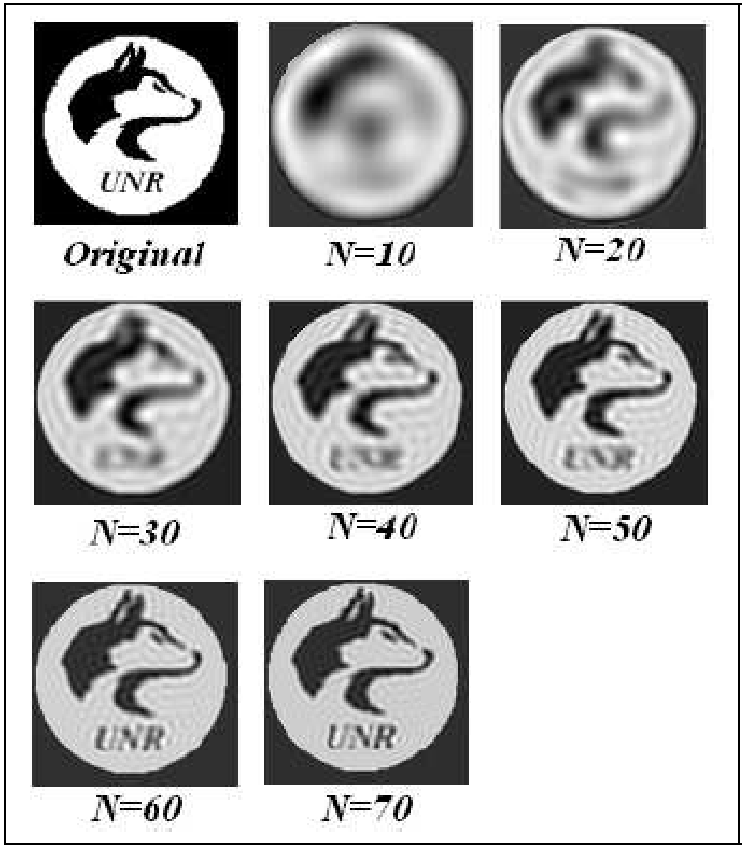

Fig. 6. Original image and reconstructions using different orders of Zernike moments.