Probabilities, Greyscales, and Histograms: Chapter 3a G&W

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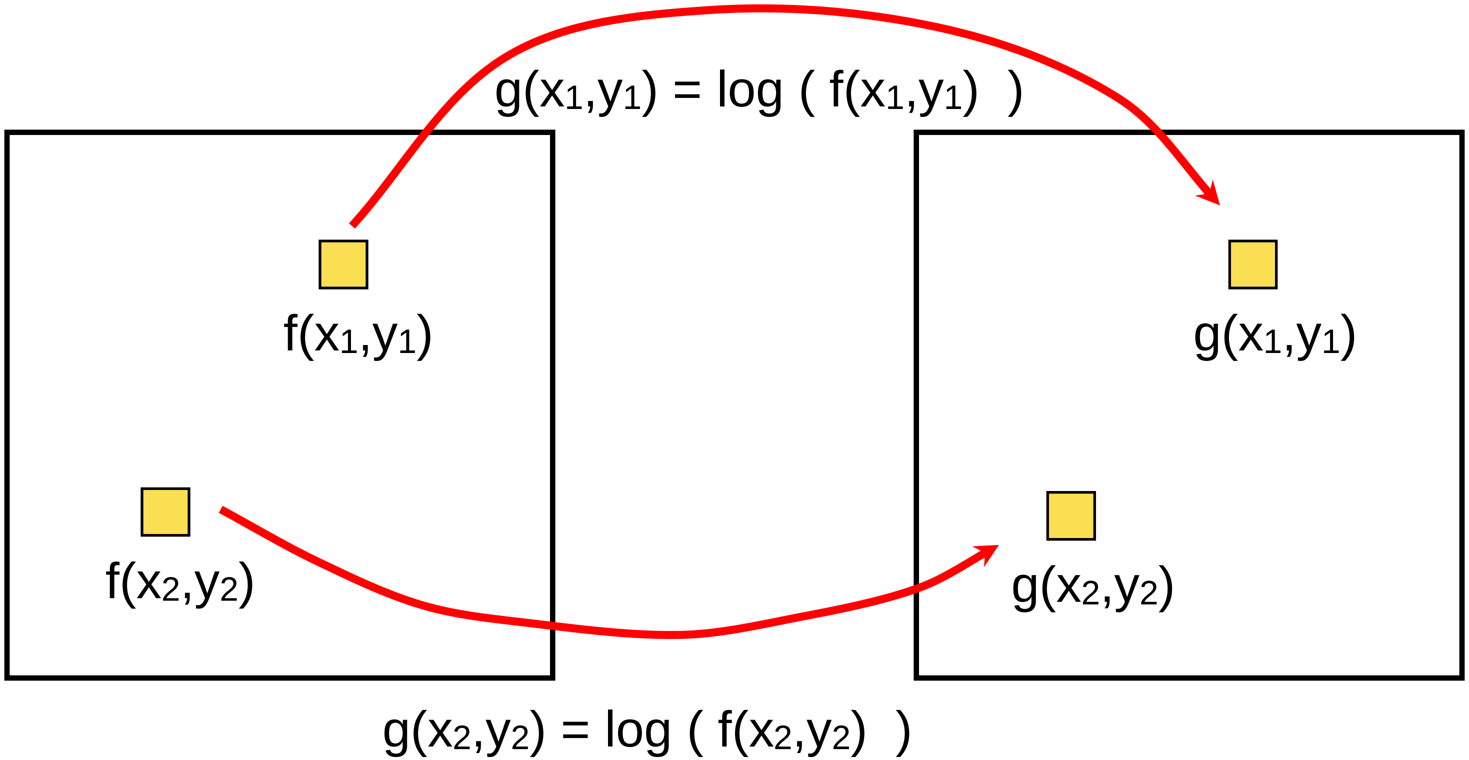
Goal

- Image intensity transformations
- Intensity transformations as mappings
- Image histograms
- Relationship btw histograms and probability density distributions
- Repetition: Probabilities

- Image segmentation via thresholding

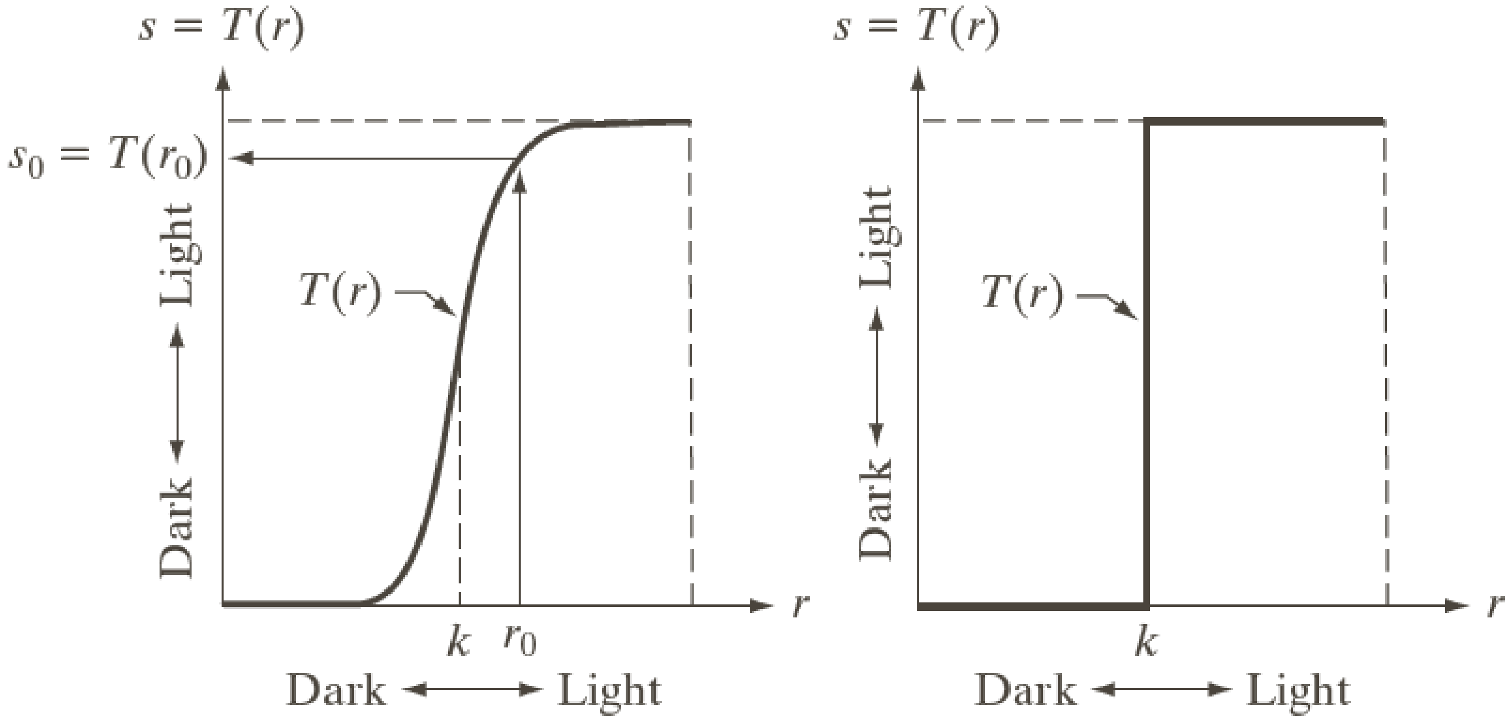
Intensity transformation example

$g(x,y) = \log(f(x,y))$



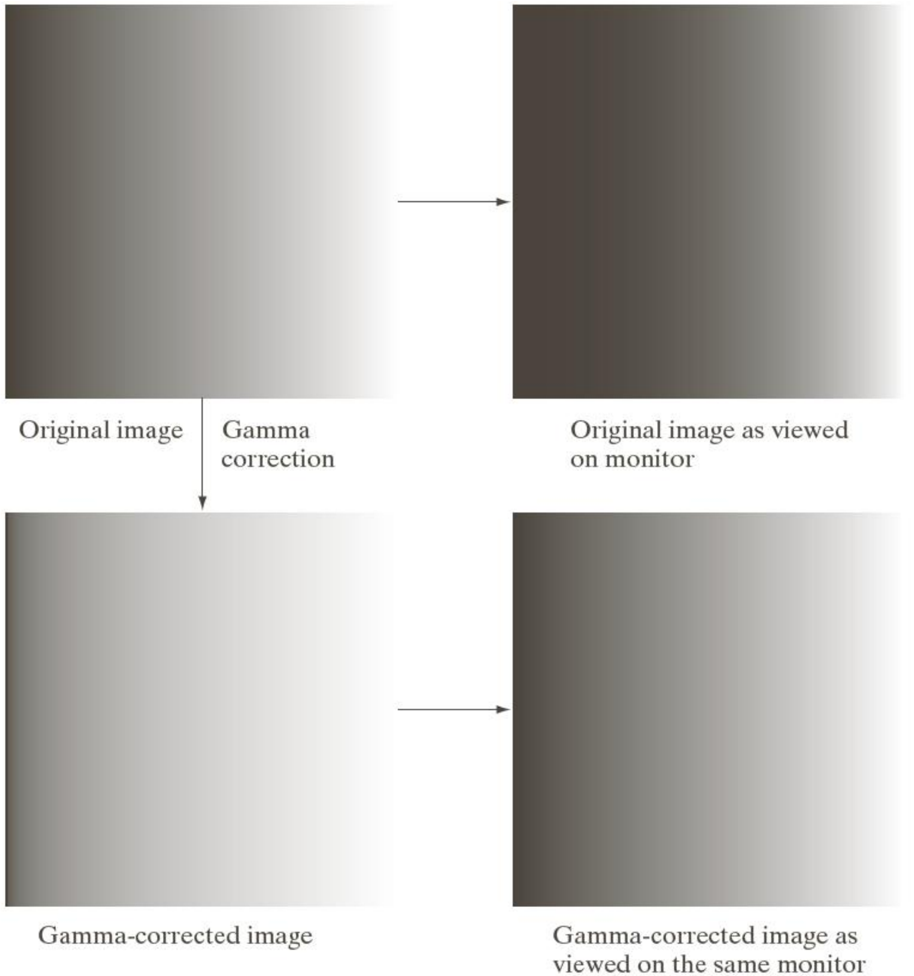
- We can **drop the (x,y)** and represent this kind of filter as an intensity transformation $s=T(r)$. In this case $s=\log(r)$
  - -s: output intensity
  - -r: input intensity

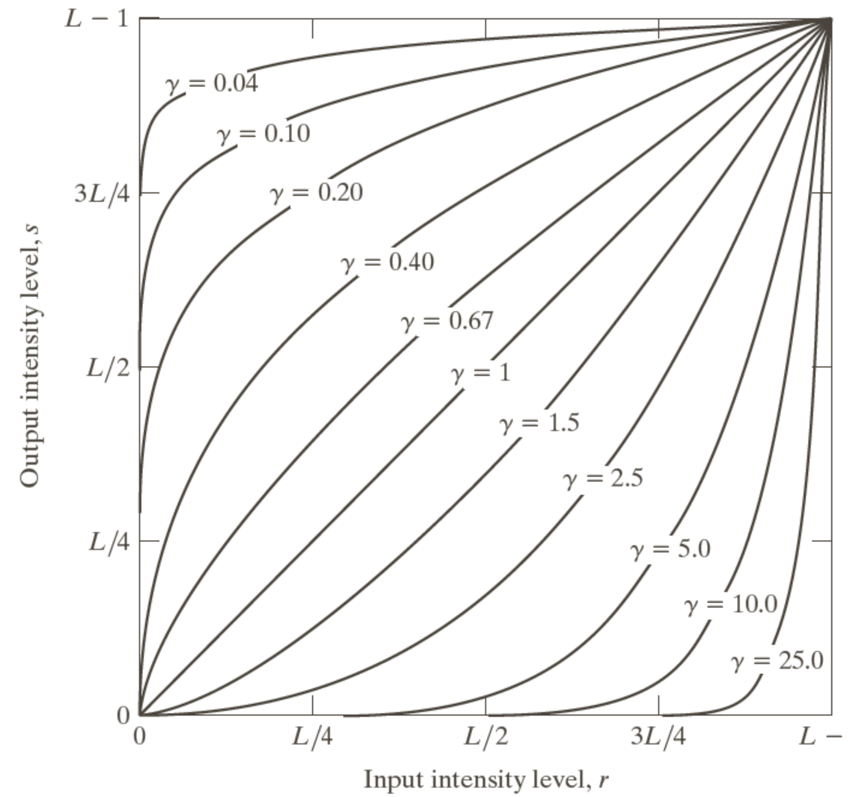
Intensity transformation



$$s = T(r)$$

Gamma correction





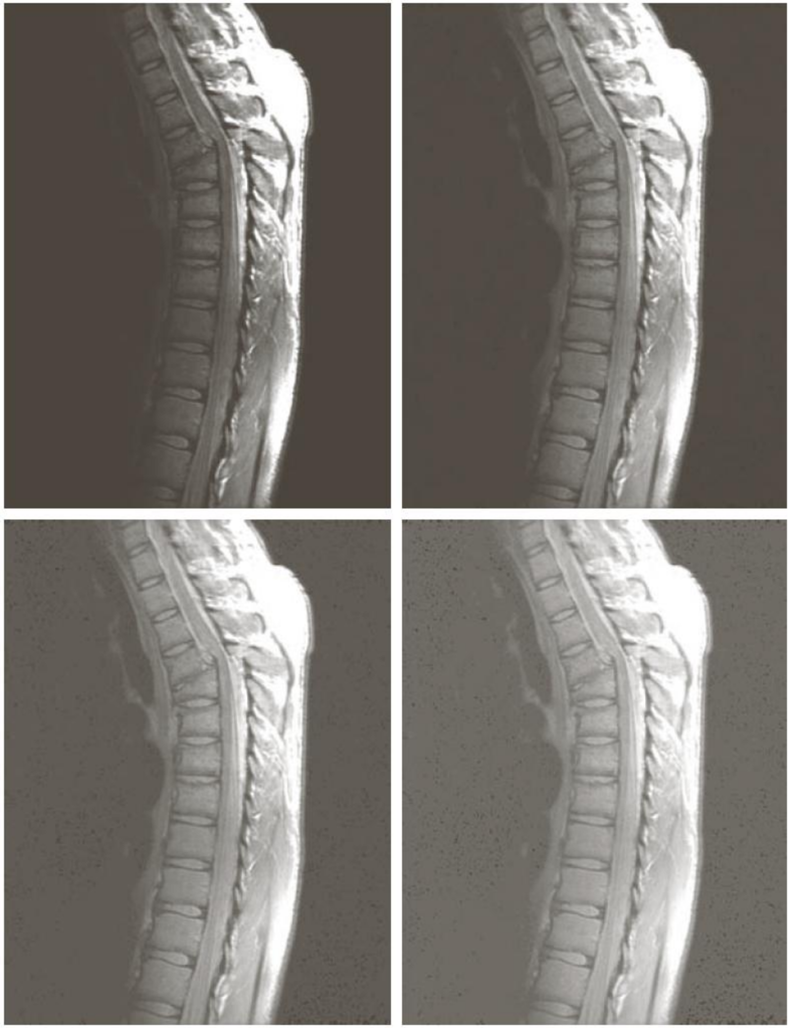
$$s = cr^{\gamma}$$

Gamma transformations

a	b
c	d

FIGURE 3.9 (a) Aerial image. (b)–(d) Results of applying the transformation in Eq. (3.2-3) with $c = 1$ and $\gamma = 3.0, 4.0,$ and 5.0, respectively. (Original image for this example courtesy of NASA.)

Gamma transformations

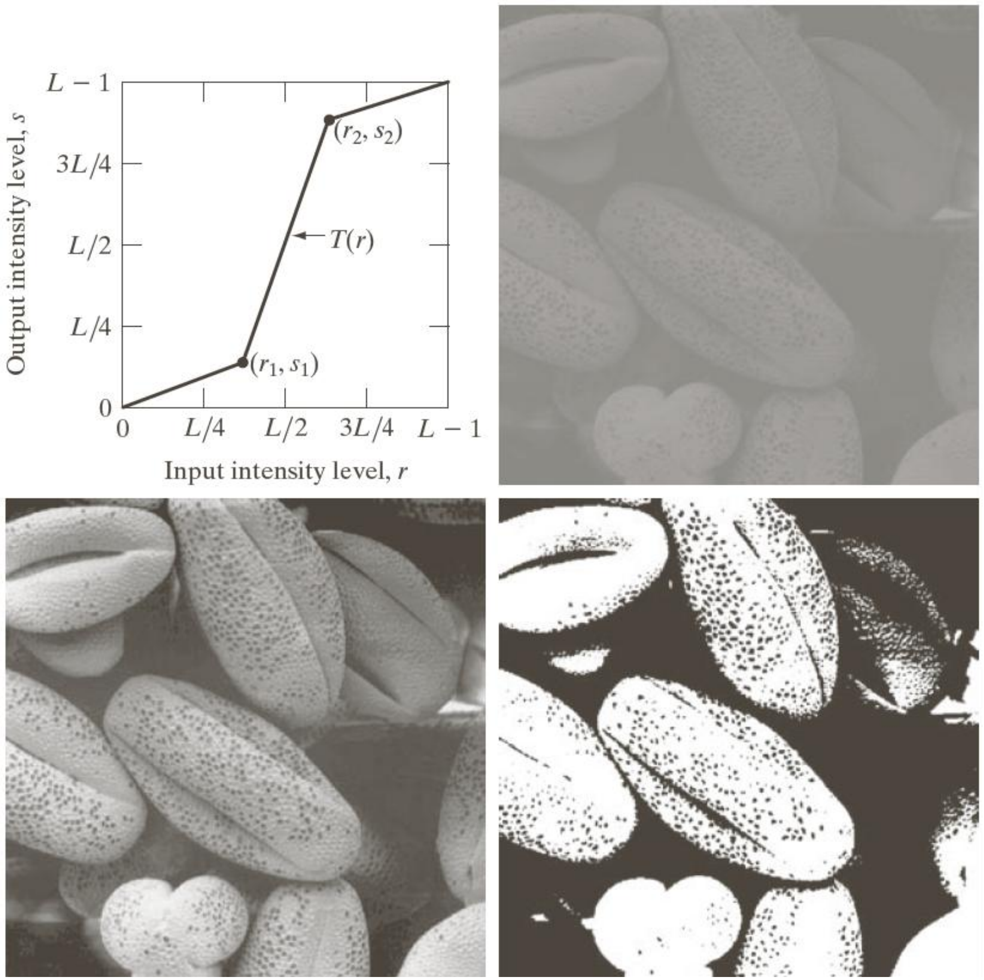


a b
c d

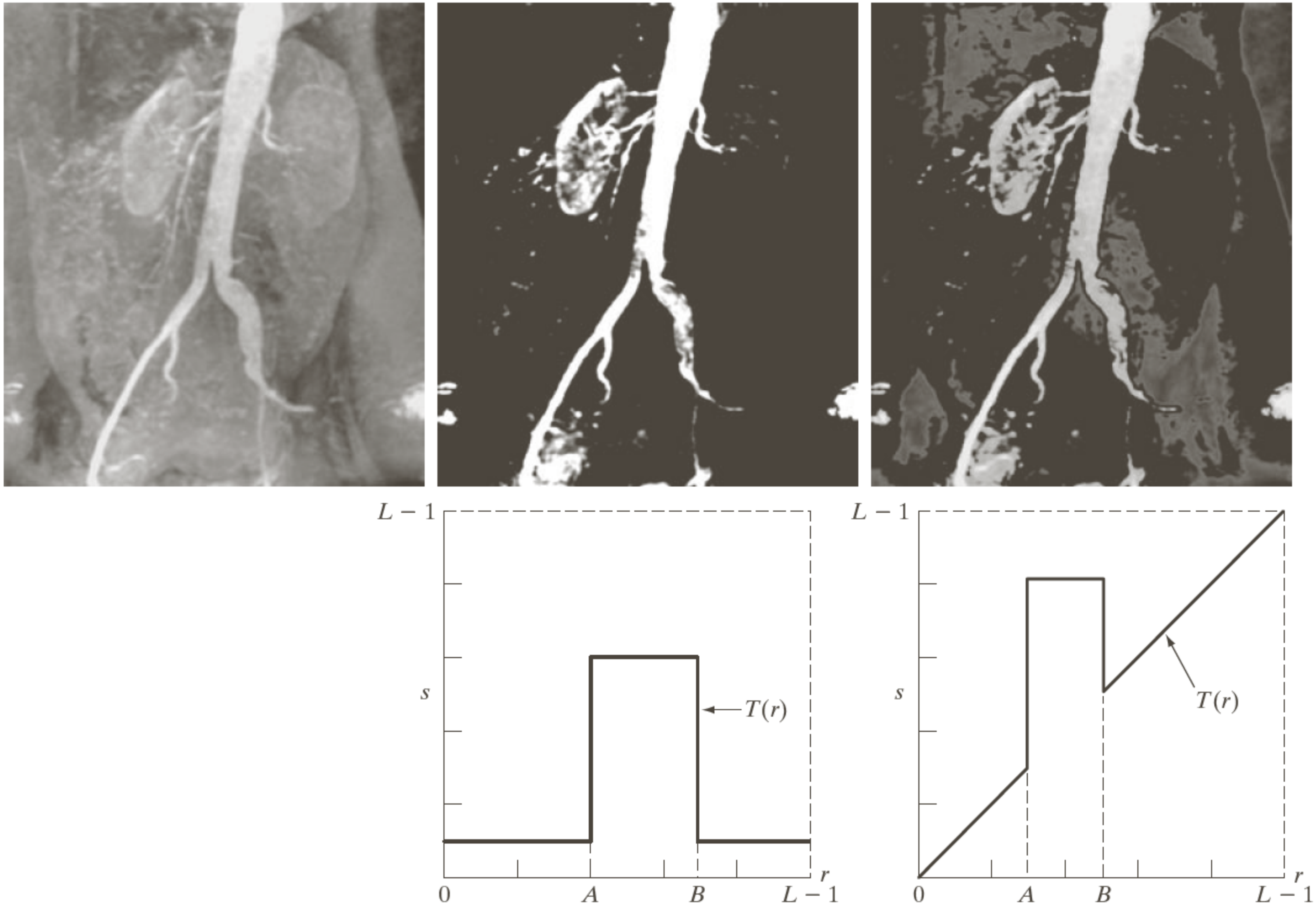
FIGURE 3.8 (a) Magnetic resonance image (MRI) of a fractured human spine. (b)–(d) Results of applying the transformation in Eq. (3.2-3) with $c = 1$ and $\gamma = 0.6, 0.4,$ and 0.3, respectively. (Original image courtesy of Dr. David R. Pickens, Department of Radiology and Radiological Sciences, Vanderbilt University Medical Center.)

Piecewise linear intensity transformation

- More control
- But also more parameters for user to specify
  - Graphical user interface can be useful

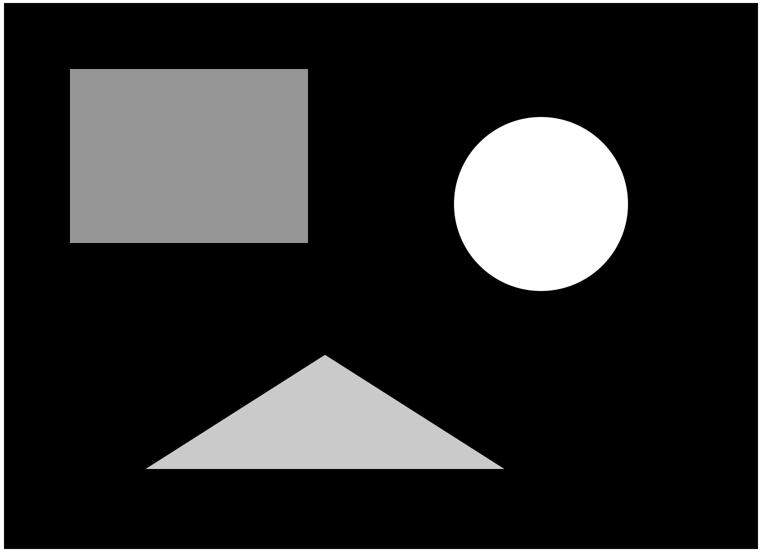


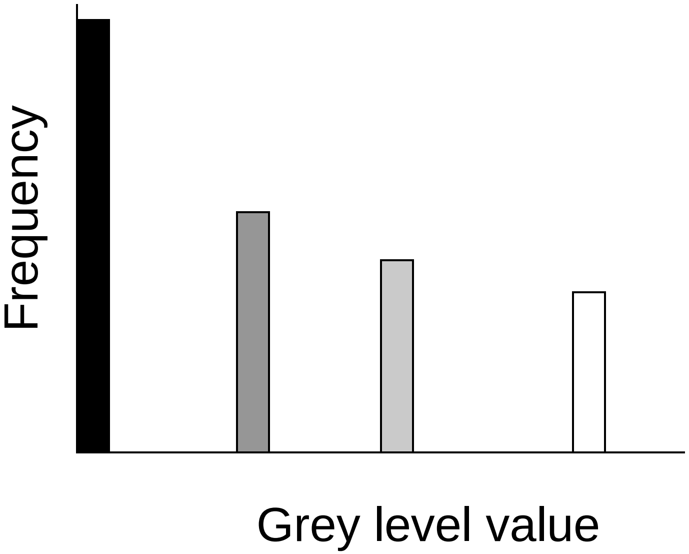
More intensity transformations



Histogram of Image Intensities

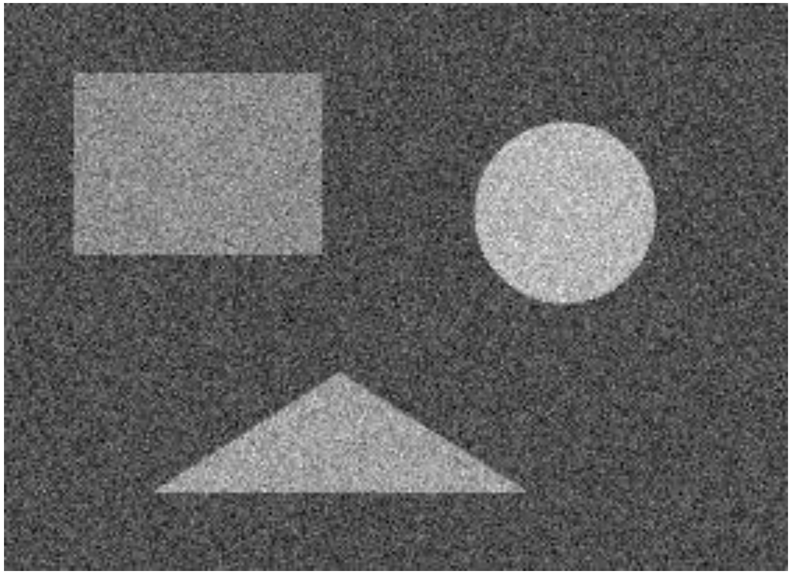
- Create bins of intensities and count number of pixels at each level
  - Normalize or not (absolute vs % frequency)

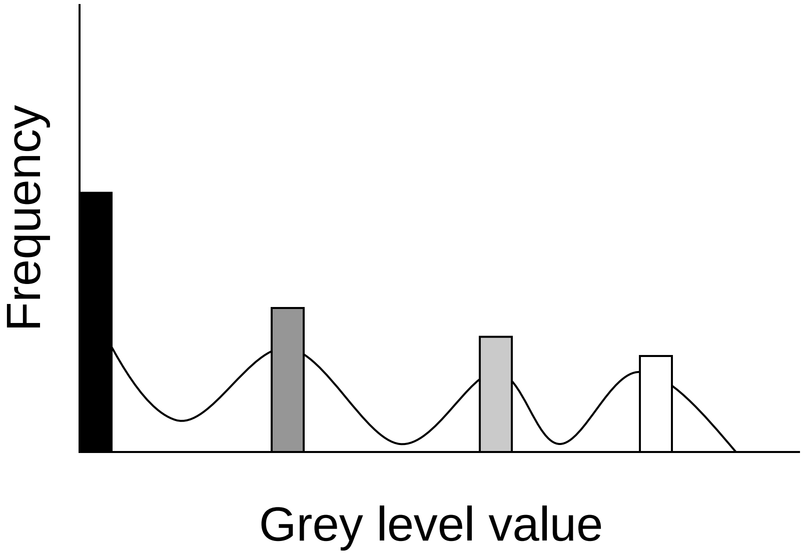




Histograms and Noise

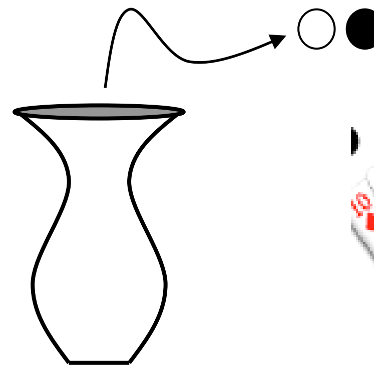
- What happens to the histogram if we add noise?
  - $g(x, y) = f(x, y) + n(x, y)$





Sample Spaces

- S = <u>Set</u> of possible outcomes of a random event
- Toy examples
  - Dice
  - Urn
  - Cards



- Probabilities

$$P(S) = 1 \qquad A \in S \Rightarrow P(A) \geq 0$$

$$P(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i) \;\; \text{where} \;\; \mathrm{A_i} \cap \mathrm{A_j} = \emptyset$$

$$\cup_{i=1}^{n} A_i = S \Rightarrow \sum_{i=1}^{n} P(A_i) = 1$$

Conditional Probabilities

- Multiple events
  - S2 = SxS Cartesian produce - sets
  - Dice - (2, 4)
  - Urn - (black, black)
- P(A|B) - probability of A in second experiment knowledge of outcome of first experiment
  - This quantifies the effect of the first experiment on the second
- P(A,B) - probability of A in second experiment and B in first experiment
- P(A,B) = P(A|B)P(B)

Independence

- $P(A|B) = P(A)$
  - The outcome of one experiment does not affect the other
- Independence -> $P(A,B) = P(A)P(B)$
- Dice
  - Each roll is unaffected by the previous (or history)
- Urn
  - Independence -> put the stone back after each experiment
- Cards
  - Put each card back after it is picked

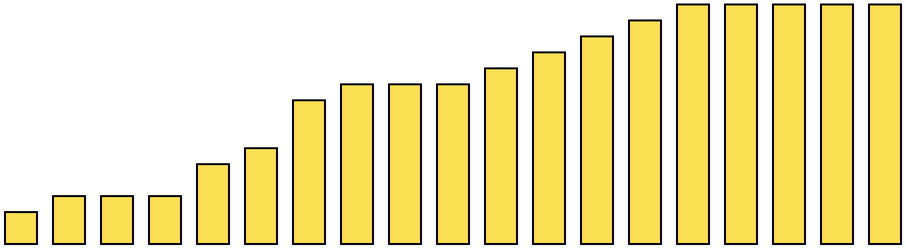
Random Variable (RV)

- Variable (number) associated with the outcome of an random experiment
- Dice
  - E.g. Assign 1-6 to the faces of dice
- Urn
  - Assign 0 to black and 1 to white (or vise versa)
- Cards
  - Lots of different schemes - depends on application
- A function of a random variable is also a random variable

Cumulative Distribution Function (cdf)

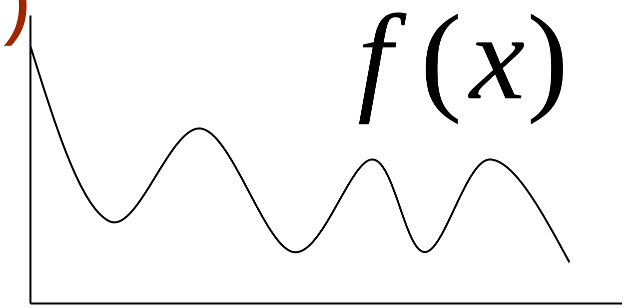
- F(x), where x is a RV
- F(-infty) = 0, F(infty) = 1
- F(x) non decreasing

$$F(x) = \sum_{i=-\infty}^{x} P(i)$$



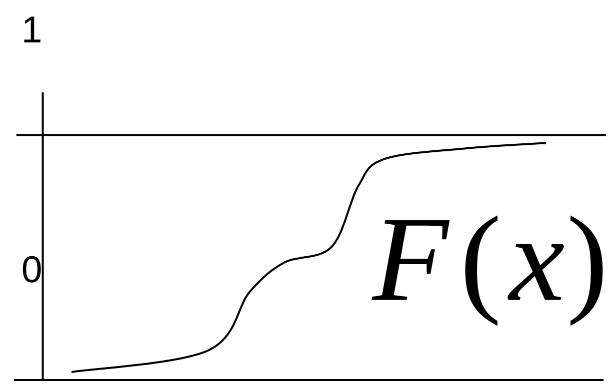
Continuous Random Variables

- f(x) is pdf (normalized to 1)
- F(x) – cdf continuous
  - – –> x is a continuous RV



$$F(x) = \int_{-\infty}^{x} f(q)dq$$

$$f(x) = \left.\frac{dF(q)}{dq}\right|_{x} = F'(x)$$



Probability Density Functions

- f(x) is called a probability density function (pdf)

$$\int_{-\infty}^{\infty} f(x) = 1 \quad f(x) \geq 0 \ \forall \ x$$

- A probability density is not the same as a probability
- The probability of a specific value as an outcome of continuous experiment is (generally) zero
  - To get meaningful numbers you must specify a range

$$P(a \leq x \leq b) = \int_{a}^{b} f(q)dq = F(b) - F(a)$$

Expected Value of a RV

$$E[x] = \sum_{i=-\infty}^{\infty} i \; p(i)$$

$$E[x] = \int_{-\infty}^{\infty} q \; f(q) \; dq$$

- Expectation is linear
  - E[ax] = aE[x] for a scalar (not random)
  - E[x + y] = E[x] + E[y]
- Other properties
  - E[z] = z ———— if z is not random

Mean of a PDF

- Mean: $E[x] = m$
  - also called "μ"
  - The mean is not a random variable–it is a fixed value for any PDF
- Variance: $E[(x - m)^2] = E[x^2] - 2E[mx] + E[m^2] = E[x^2] - m^2 = E[x^2] - E[x]^2$
  - also called "σ^2"
  - Standard deviation is σ
  - If a distribution has zero mean then: $E[x^2] = \sigma^2$

Sample Mean

- Run an experiments
  - Take N samples from a pdf (RV)
  - Sum them up and divide by N
- Let M be the result of that experiment
  - M is a random variable

$$M = \frac{1}{N} \sum_{i=1}^{N} x_i$$

$$E[M] = E[\frac{1}{N} \sum_{i=1}^{N} x_i] = \frac{1}{N} \sum_{i=1}^{N} E[x_i] = m$$

Sample Mean

- How close can we expect to be with a sample mean to the true mean?
- Define a new random variable: D = (M - m)^{2}.
  - Assume independence of sampling process

$$D = \frac{1}{N^2} \sum_i x_i \sum_j x_j - \frac{1}{N} 2m \sum_i x_i + m^2$$

Independence –> E[xy] = E[x]E[y]

$$\begin{aligned} e[D] &= \tfrac{1}{N^2} E[\textstyle\sum_i x_i \sum_j x_j] - \tfrac{1}{N} 2mE[\textstyle\sum_i x_i] + m^2 \\ &= \tfrac{1}{N^2} E[\textstyle\sum_i x_i \sum_j x_j] - m^2 \end{aligned}$$

Number of terms off diagonal

$$\frac{1}{N^2} E[\sum_i x_i \sum_j x_j] = \frac{1}{N^2} \sum_i E[x_i^2] + \frac{1}{N^2} \sum_i \sum_j E[x_i x_j] = \frac{1}{N} \sum_i E[x^2] + \frac{N(N-1)}{N^2} m^2$$

$$E[D] = \frac{1}{N} E[x^2] + \frac{N(N-1)}{N^2} m^2 - \frac{N^2}{N^2} m^2 = \frac{1}{N} \left(E[x^2] - m^2 \right) = \frac{1}{N} \sigma^2$$

Root mean squared difference between true mean and sample mean is stdev/sqrt(N).
As number of samples –> infty, sample mean –> true mean.

Application: Noisy Images

- Imagine N images of the same scene with random, independent, zero-mean noise added to each one
  - Nuclear medicine–radioactive events are random
  - Noise in sensors/electronics
- Pixel value is s+n

True pixel value

Random noise:

•Independent from one image to the next

•Variance = σ

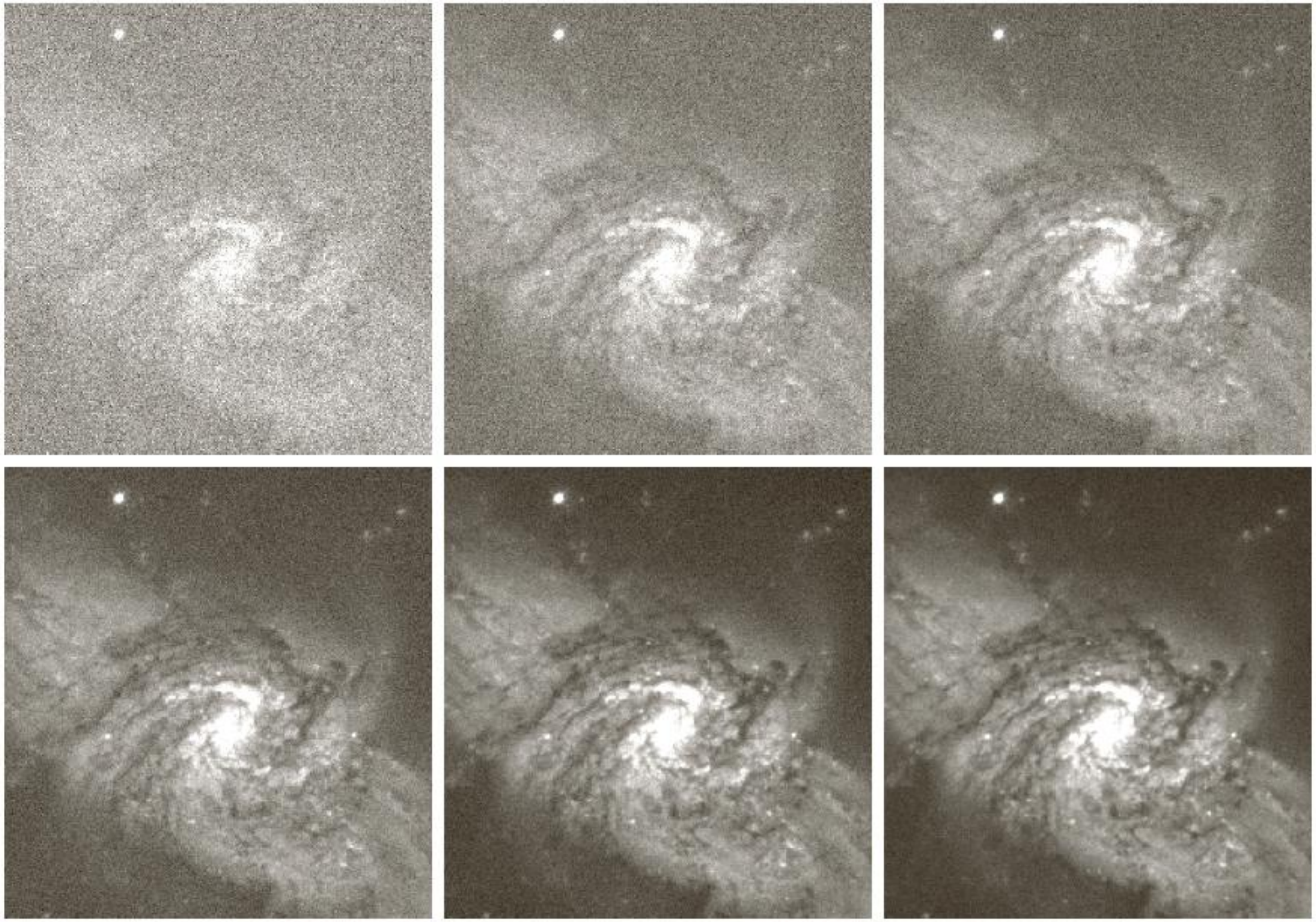
Application: Noisy Images

- If you take multiple images of the same scene you have
  - $s_i = s + n_i$
  - $S = (1/N)\ \Sigma s_i = s + (1/N)\ \Sigma n_i$
  - $E[(S - s)^2] = (1/N)\ E[n_i^{\,2}] = (1/N)\ E[n_i^{\,2}] - (1/N)\ E[n_i]^2 = (1/N)\sigma^2$
  - Expected **root mean squared error** is σ/sqrt(N)

Zero mean

- Application:
  - Digital cameras with large gain (high ISO, light sensitivity)
  - Not necessarily random from one image to next
    - Sensors CCD irregularity
  - How would this principle apply

Averaging Noisy Images Can Improve Quality

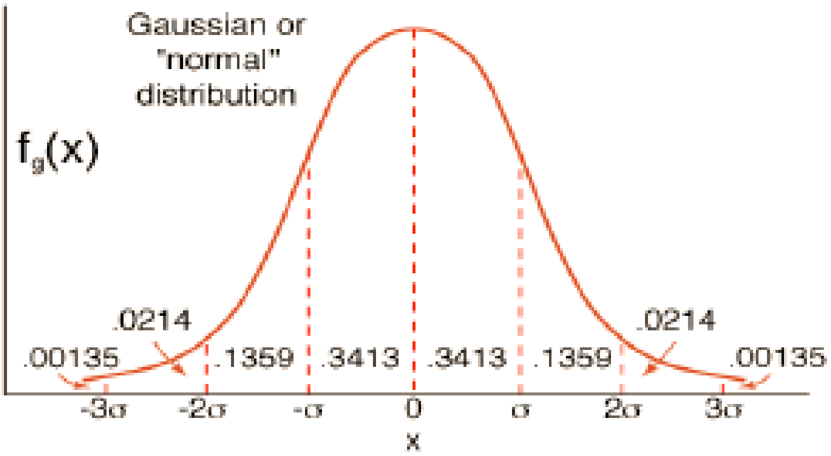


a b c
d e f

FIGURE 2.26 (a) Image of Galaxy Pair NGC 3314 corrupted by additive Gaussian noise. (b)–(f) Results of averaging 5, 10, 20, 50, and 100 noisy images, respectively. (Original image courtesy of NASA.)

Gaussian Distribution

- “Normal” or “bell curve”
- Two parameters: μ - mean, σ – standard deviation



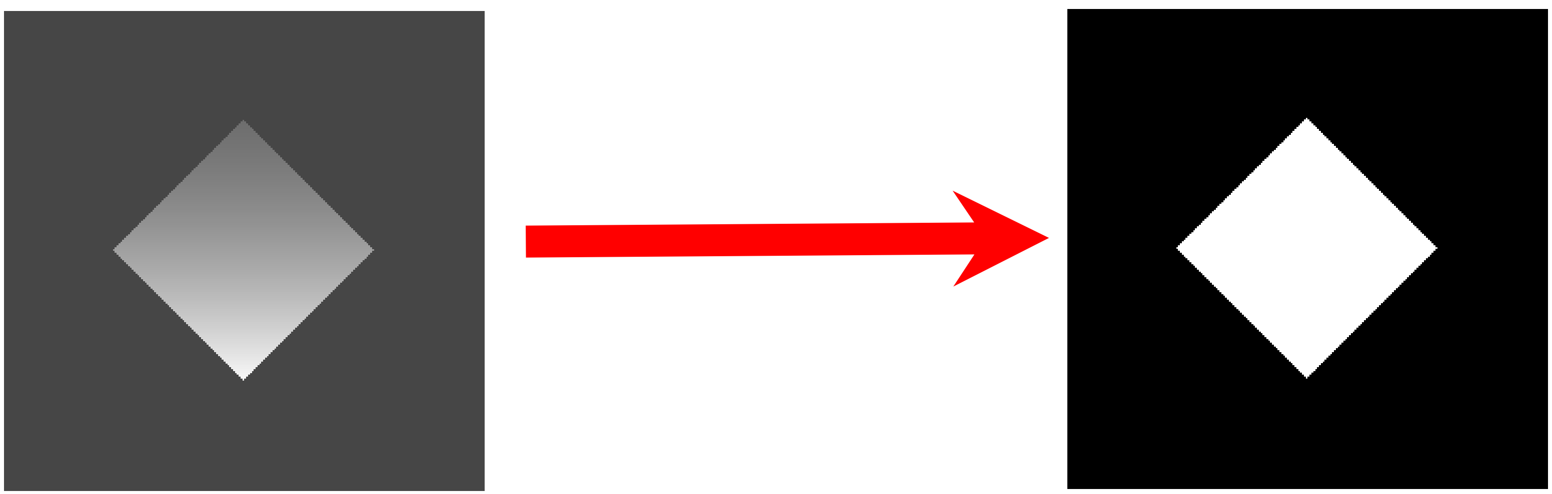
$$\frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Gaussian Properties

- Best fitting Guassian to some data is gotten by mean and standard deviation of the samples
- Occurrence
  - Central limit theorem: result from lots of random variables
  - Nature (approximate)
    - Measurement error, physical characteristic, physical phenomenon
    - Diffusion of heat or chemicals

What is image segmentation?

- Image segmentation is the process of subdividing an image into its constituent regions or objects.
- Example segmentation with two regions:

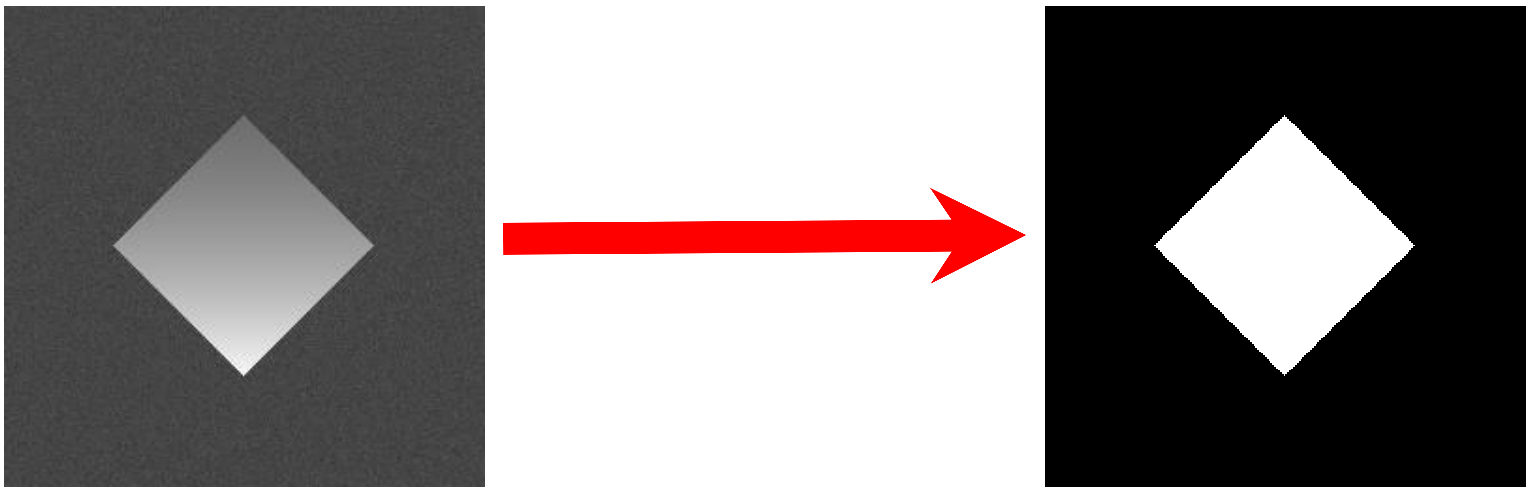


Input image
intensities 0-255

Segmentation output
0 (background)
1 (foreground)

Thresholding

$$g(x,y) = \begin{cases} 1 & if \quad f(x,y) > T \\ 0 & if \quad f(x,y) \leq T \end{cases}$$

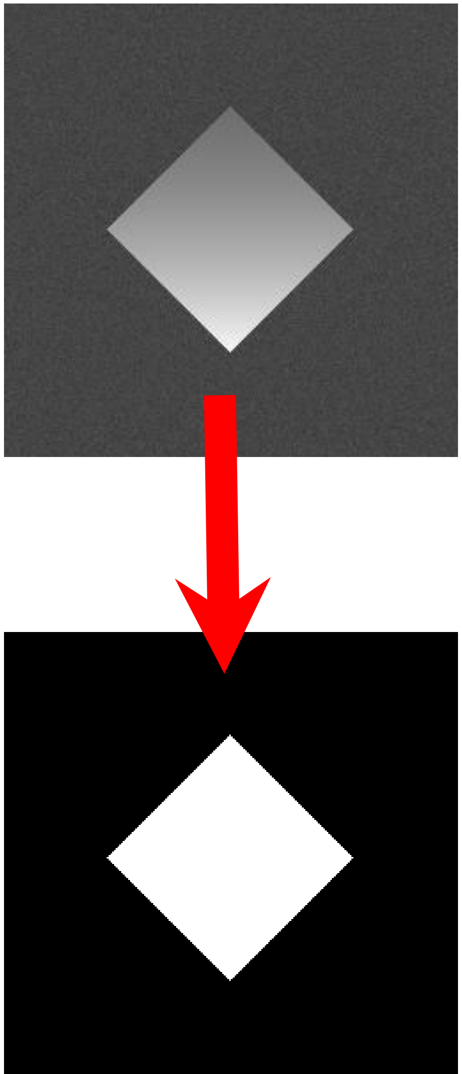


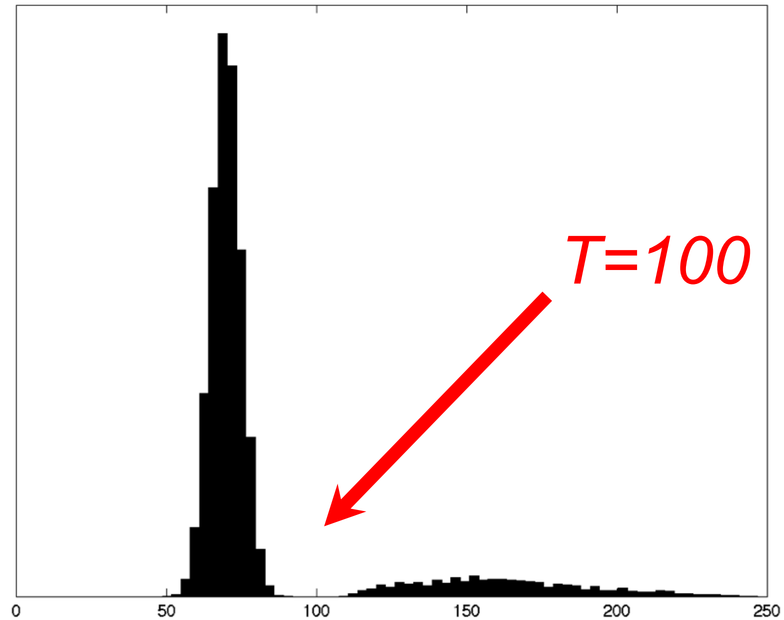
Input image f(x,y)
intensities 0-255

Segmentation output g(x,y)
0 (background)
1 (foreground)

- How can we choose T?
  - Trial and error
  - Use the histogram of f(x,y)

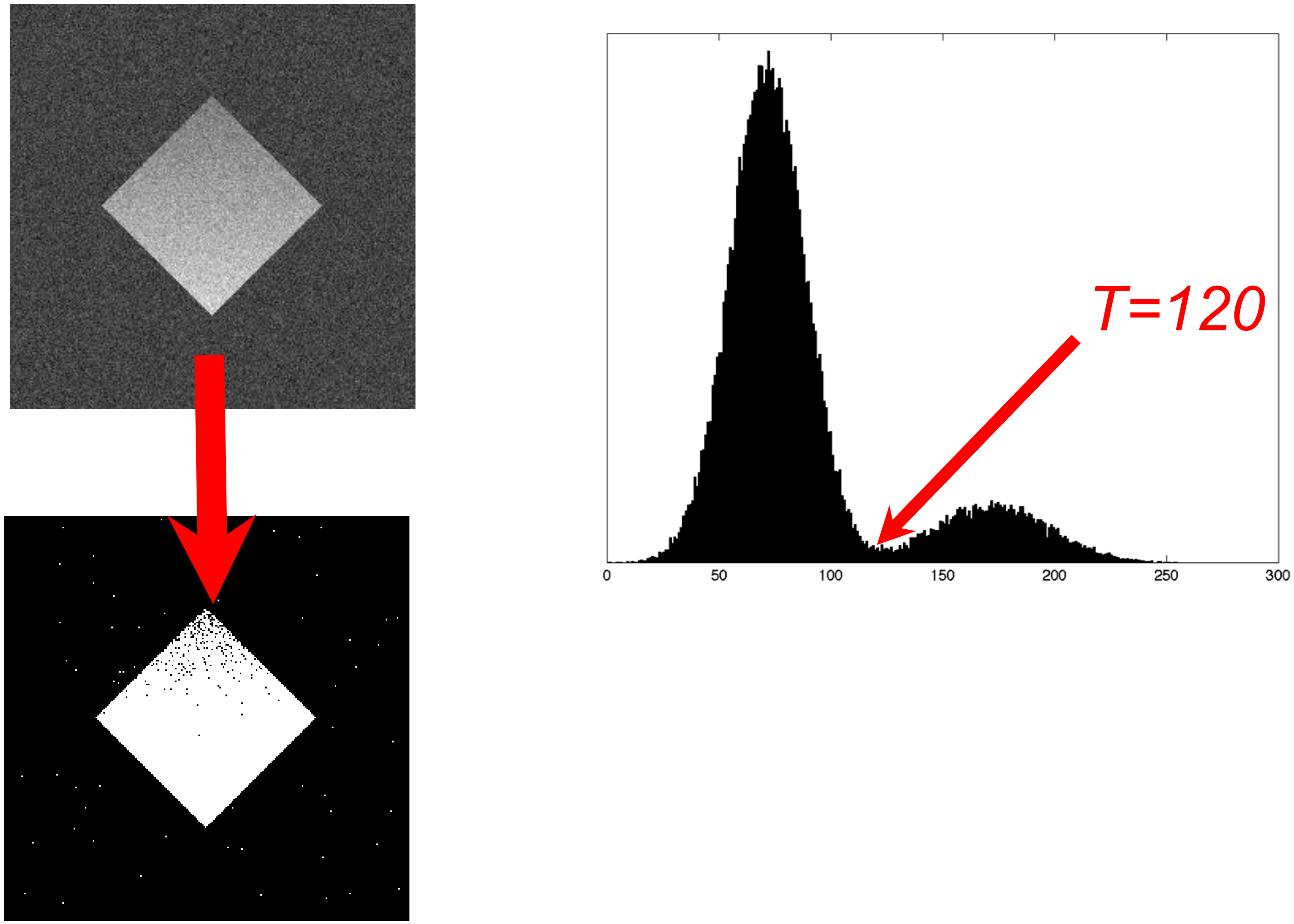
Choosing a threshold



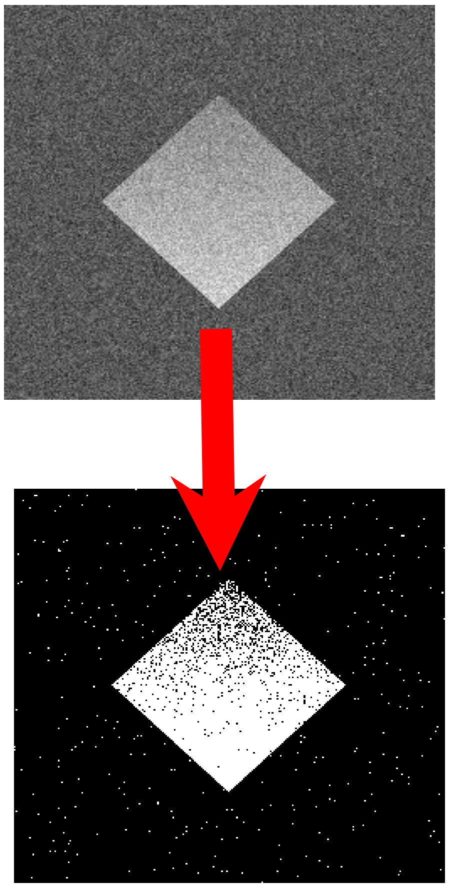


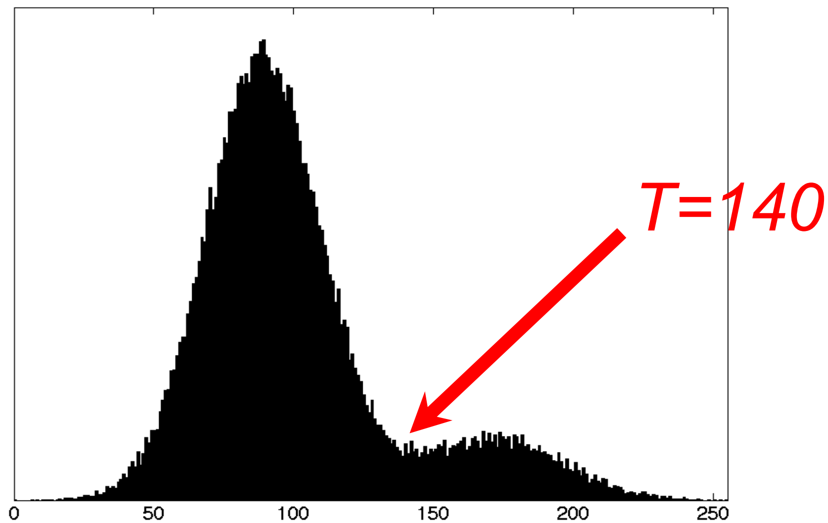
Histogram

Role of noise



Low signal-to-noise ratio

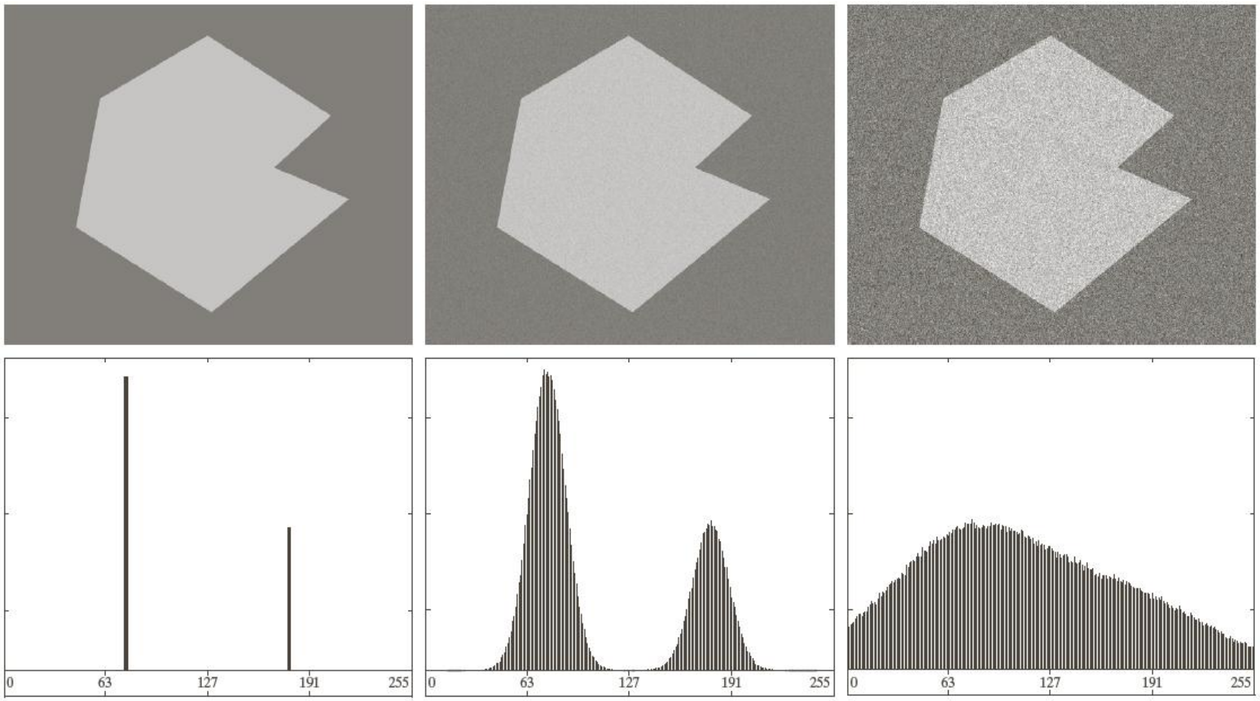




Effect of noise on image histogram

Images

Histograms

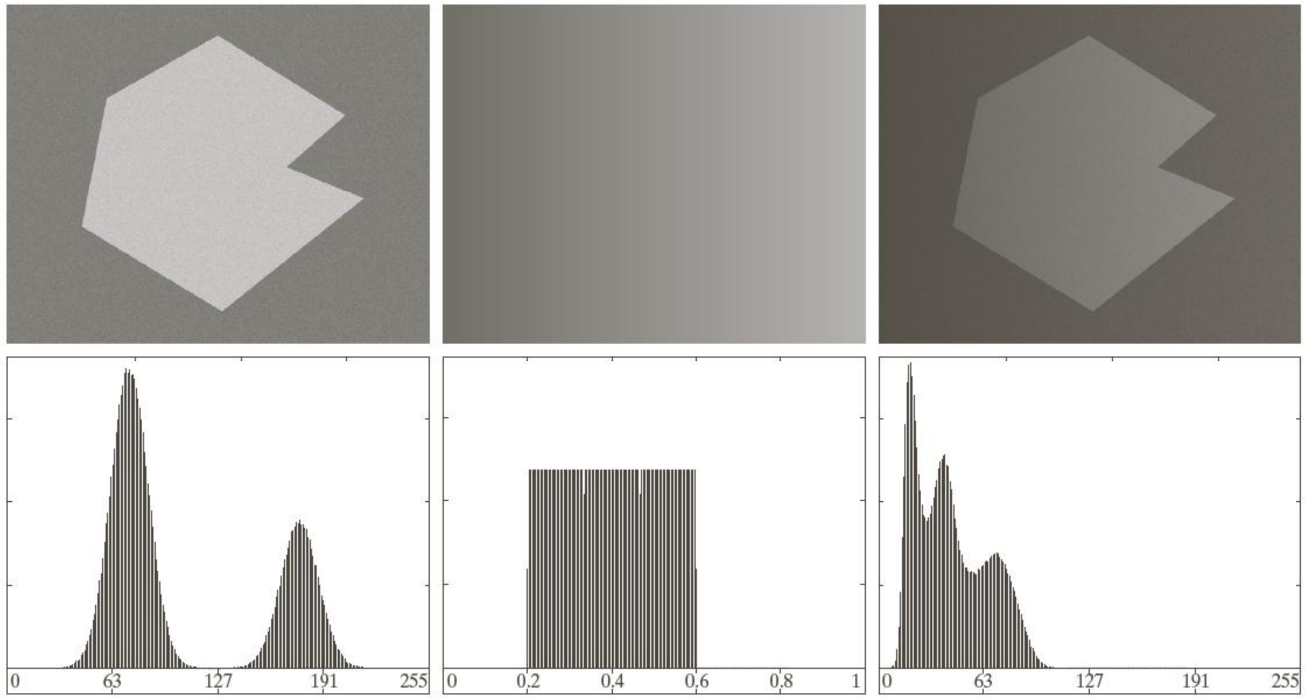


No noise   With noise   More noise

Effect of illumination on histogram

Images

Histograms



f x g = h

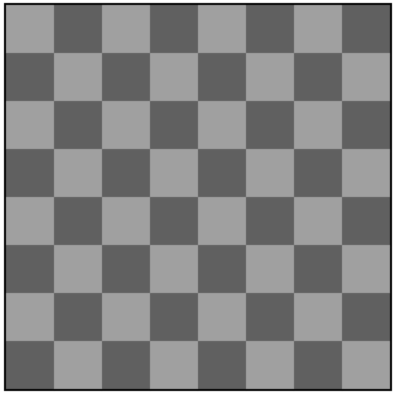
Original image

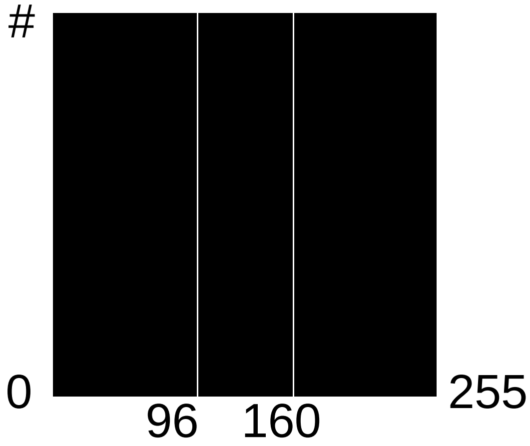
Illumination image

Final image

Histogram of Pixel Intensity Distribution

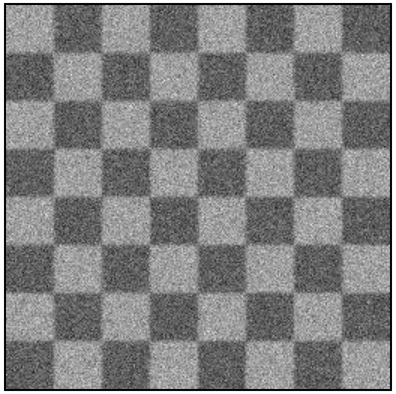
Histogram: Distribution of intensity values $p(v)$
(count #pixels for each intensity level)

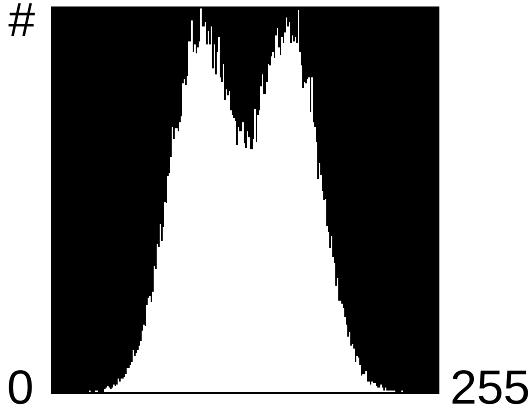




Checkerboard with values 96 and 160.

Histogram:
- horizontal: intensity
- vertical: # pixels

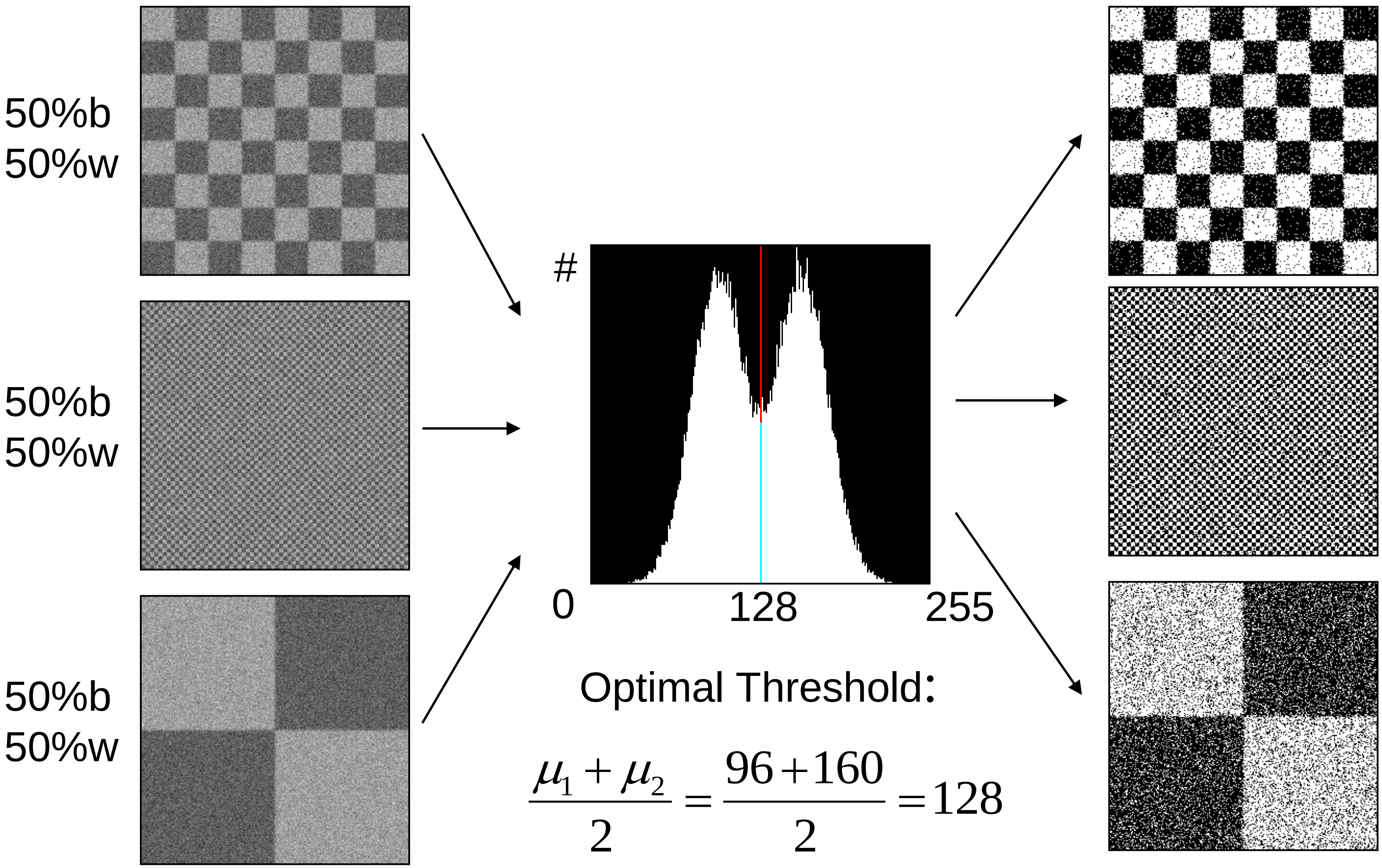




Checkerboard with additive Gaussian noise (sigma 20).

Regions: 50%b,50%w

Classification by Thresholding



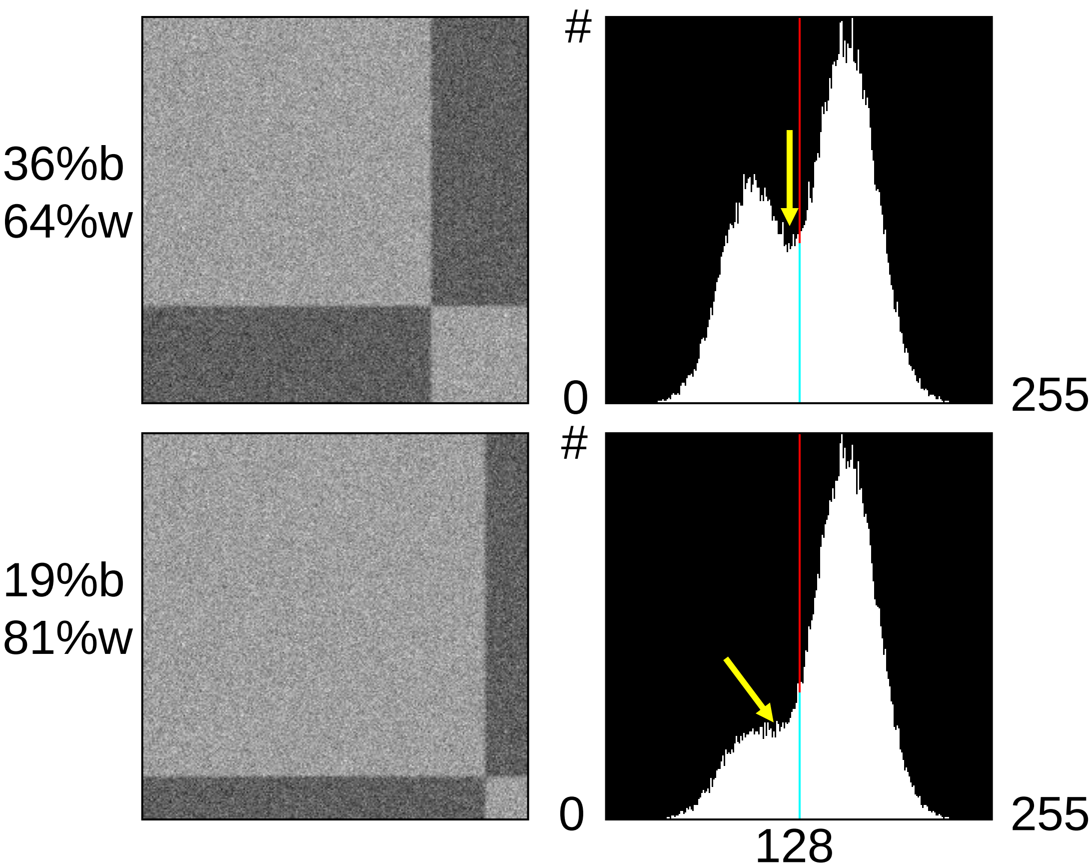
Optimal Threshold:

$$\frac{\mu_1 + \mu_2}{2} = \frac{96 + 160}{2} = 128$$

Important!

- Histogram does not represent image structure such as regions and shapes, but only distribution of intensity values
- Many images share the same histogram

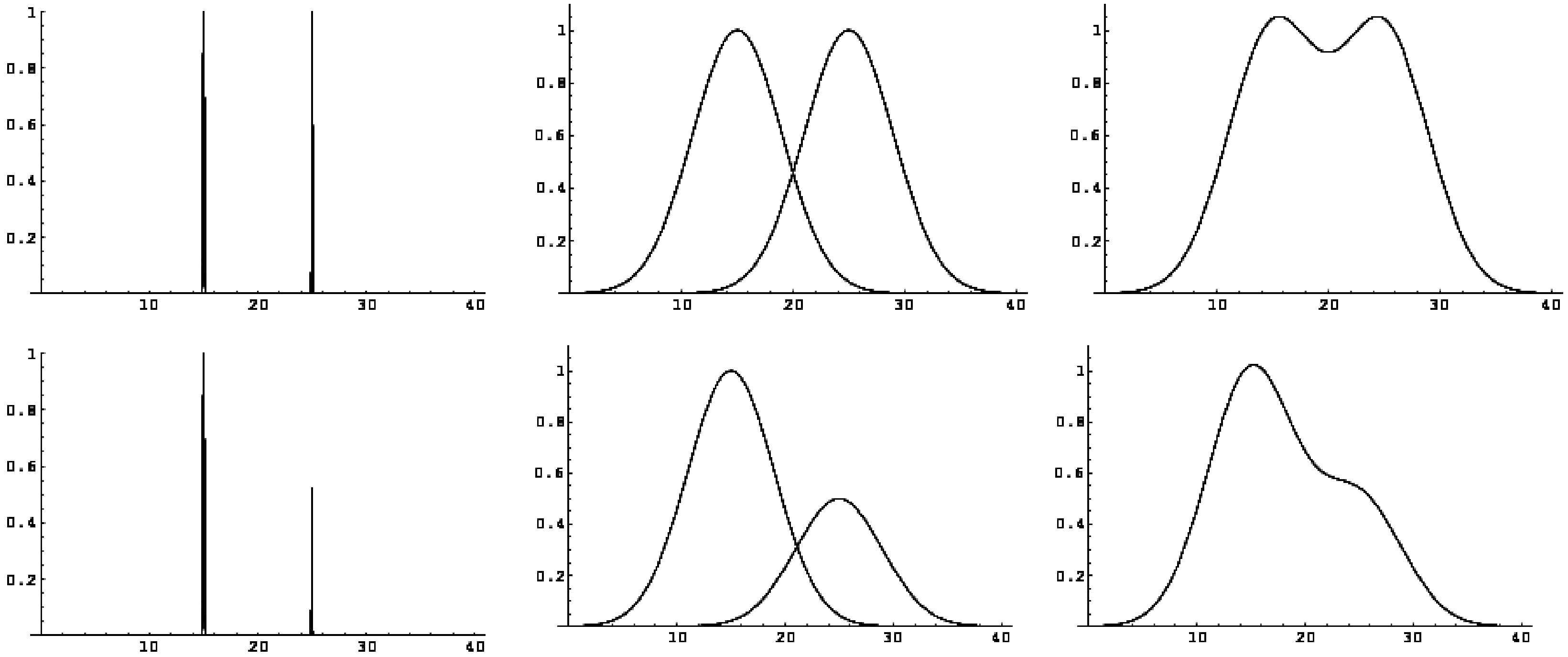
Is the histogram suggesting the right threshold?



Proportions of bright and dark regions are different ⇒ Peak presenting bright regions becomes dominant.

Threshold value 128 does not match with valley in distribution.

Histogram as Superposition of PDF's
(probability density functions)



Regions with
2 brightness levels,
different proportions

Corruption with
Gaussian noise,
individual distributions

Histogram:
Superposition of
distributions

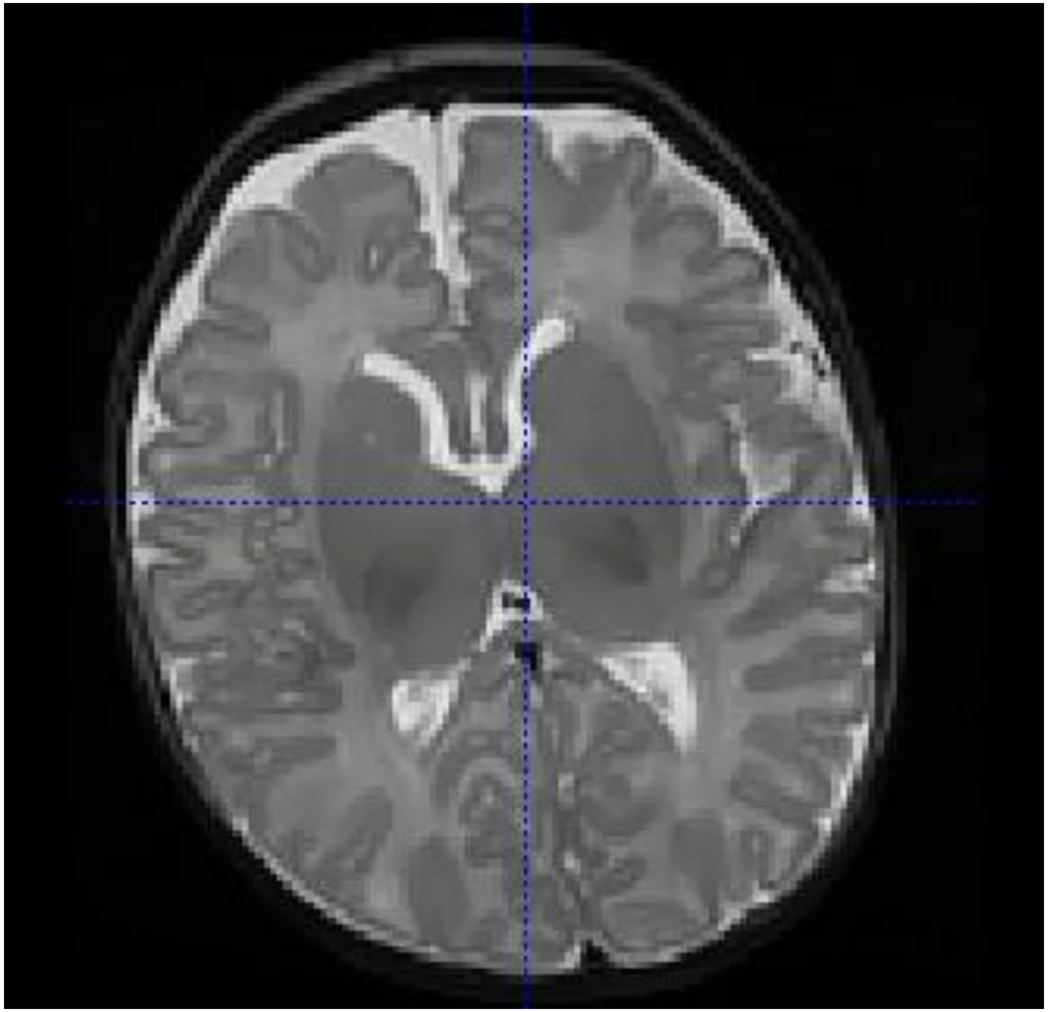
Gaussian Mixture Model

$$hist = a_1 G(\mu_1, \sigma_1) + a_1 G(\mu_1, \sigma_2)$$

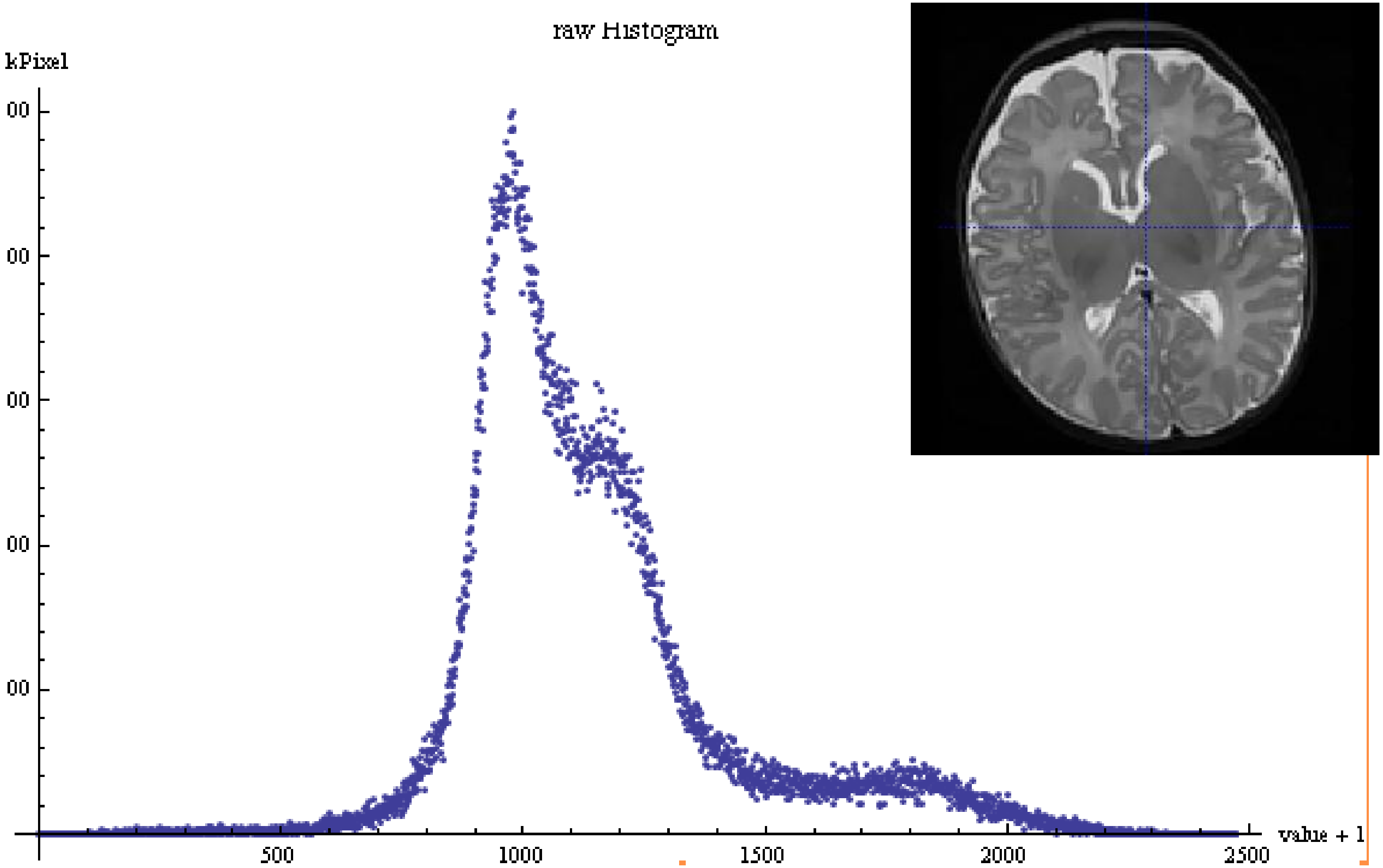
$$more\ general\ with\ k\ classes:$$

$$hist = \sum_k a_k G(\mu_k, \sigma_k)$$

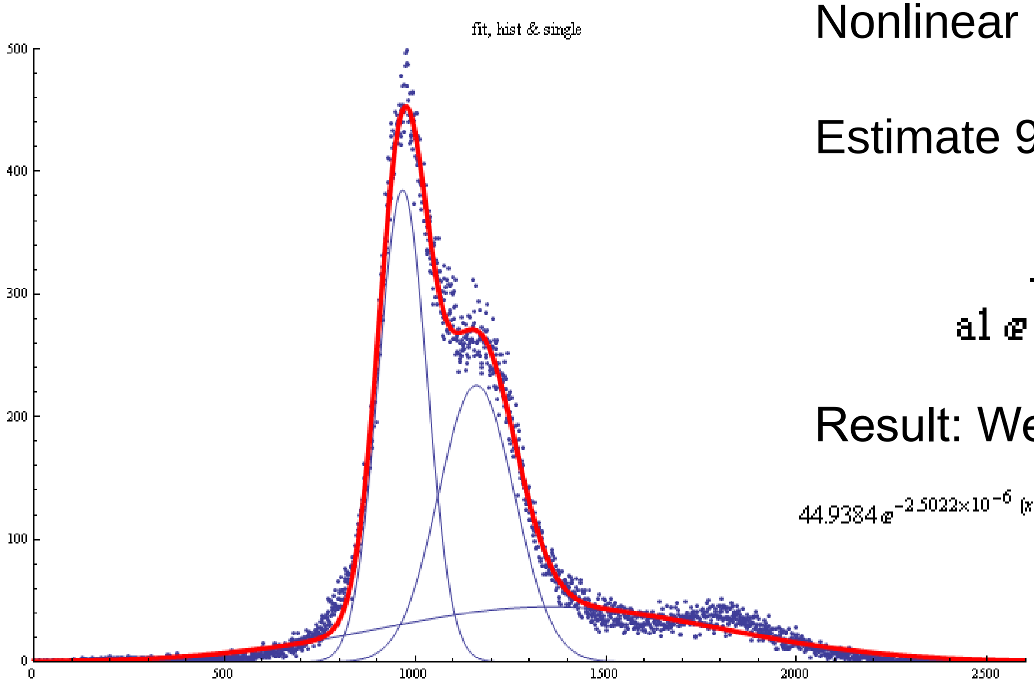
Example: MRI



Example: MRI


raw Histogram
kPixel
00
00
00
00
00
500
1000
1500
2000
2500
value + 1

Fit with 3 weighted Gaussians



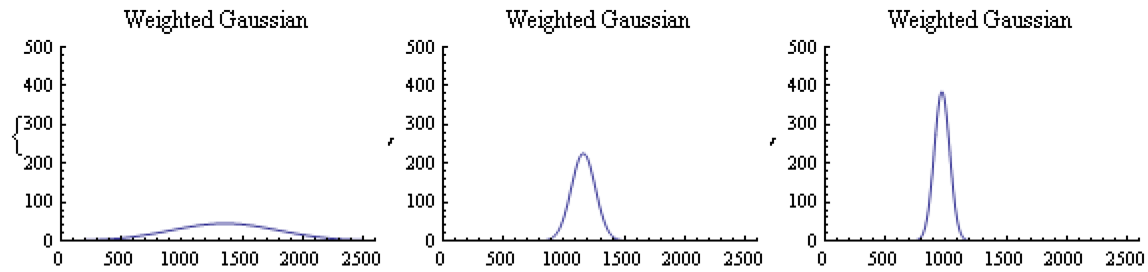
Nonlinear optimization

Estimate 9 parameters for best fit:

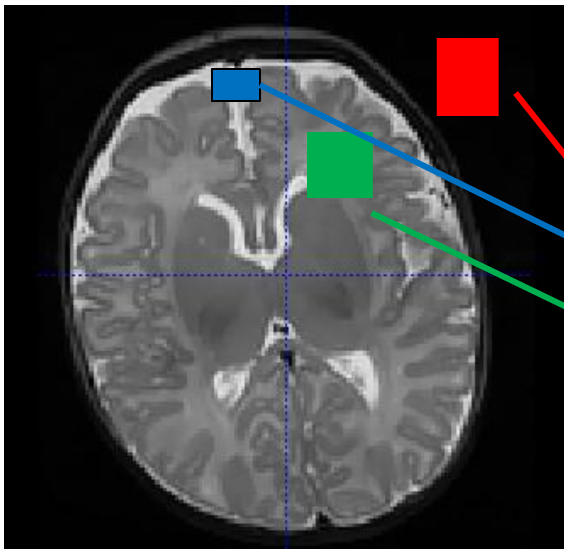
$$a1\, e^{-\frac{(x-my1)^2}{2\, sig1^2}} + a2\, e^{-\frac{(x-my2)^2}{2\, sig2^2}} + a3\, e^{-\frac{(x-my3)^2}{2\, sig3^2}}$$

Result: Weighted sum of Gaussians (pdf's):

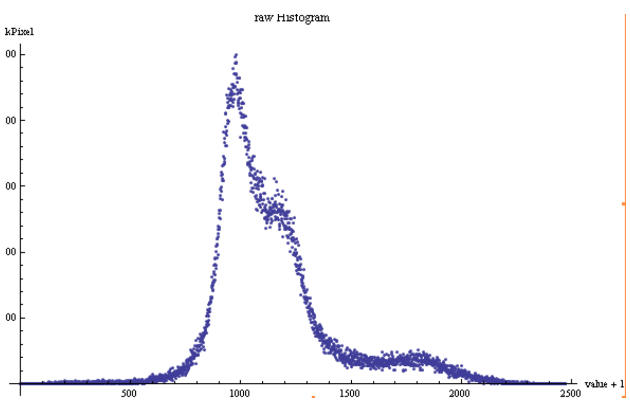
$$44.9384\, e^{-2.5022\times 10^{-6}\,(x-1353.63)^2} + 225.575\, e^{-0.0000503733\,(x-1160.5)^2} + 384.58\, e^{-0.000122748\,(x-967.112)^2}$$

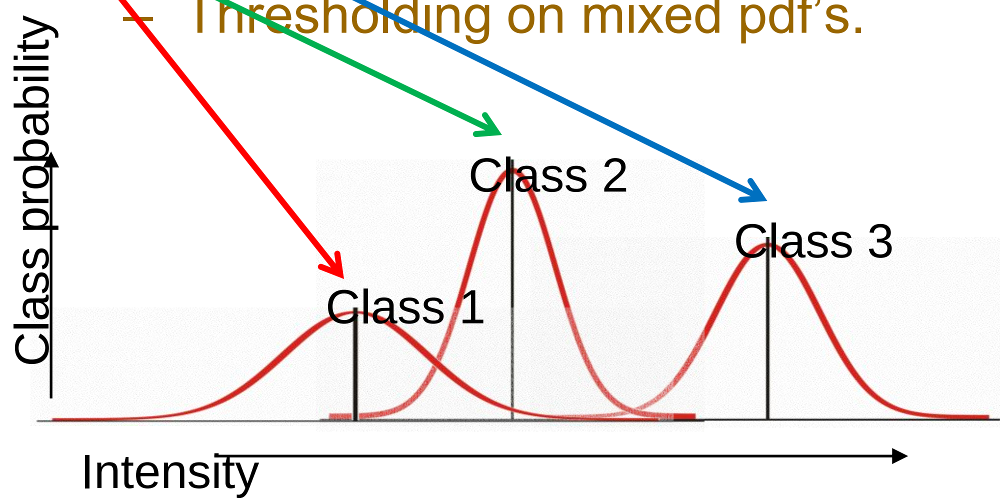


Segmentation: Learning pdf's



- We learned: histogram can be misleading due to different size of regions.
- **Solution**:
  - Estimate class-specific pdf's via training (or nonlinear optimization)
  - Thresholding on mixed pdf's.





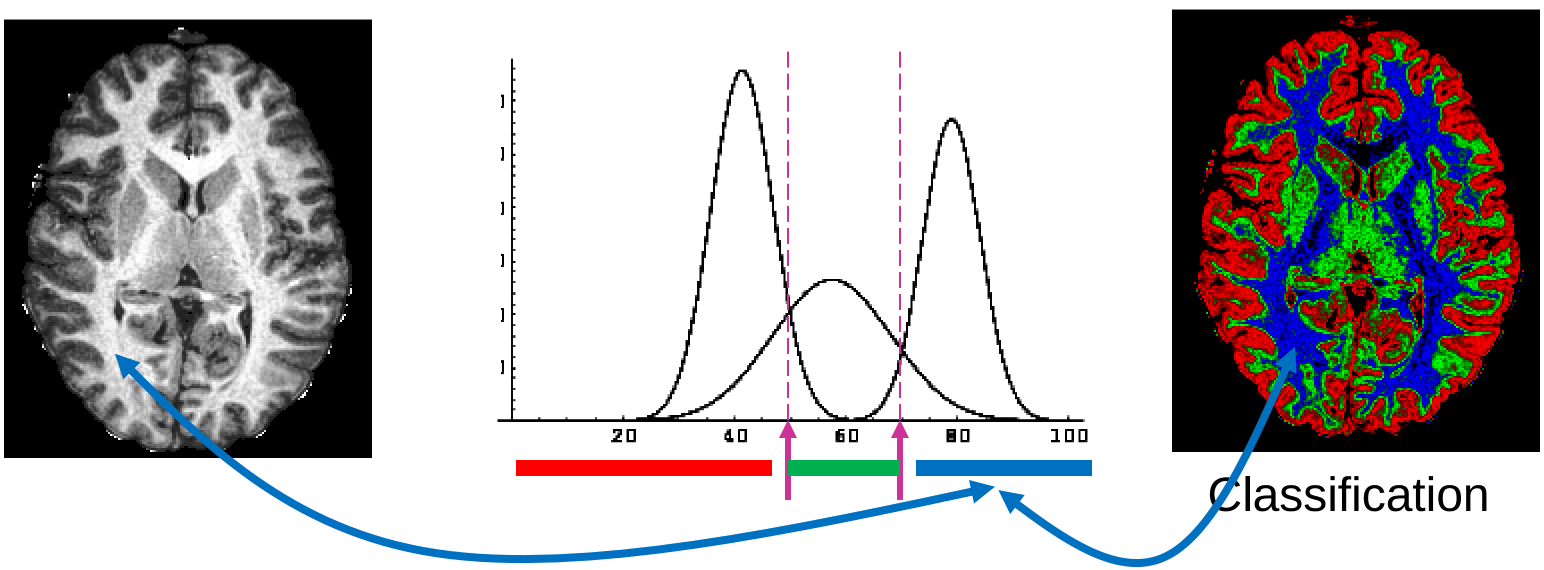
Segmentation: Learning pdf’s

set of pdf's:

$$G_k(\mu_k, \sigma_k \mid k), \quad (k = 1, \ldots, n)$$

calculate thresholds

assign pixels to categories

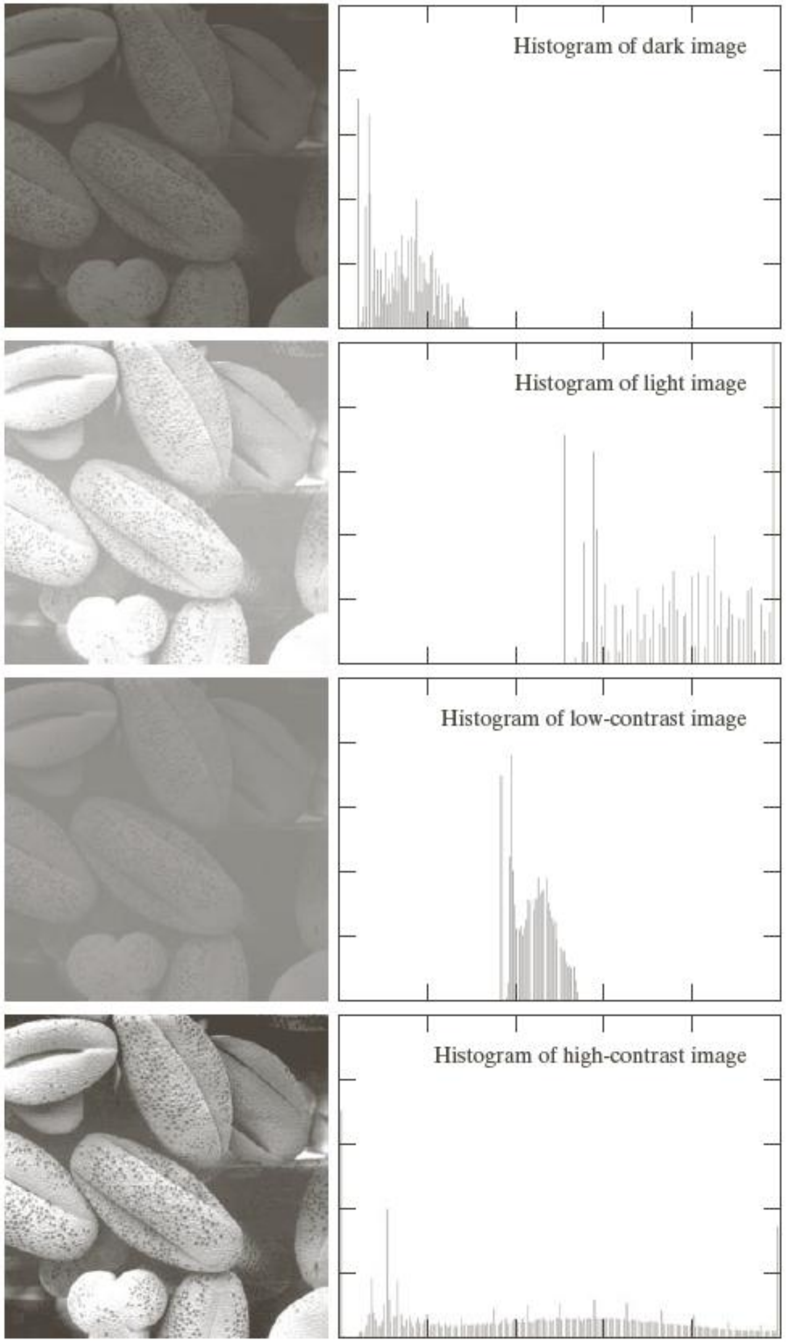


Histogram Processing and Equalization

- Notes

Histograms

- $h(r_k) = n_k$
  - Histogram: number of times intensity level r_k appears in the image
- $p(r_k) = n_k/NM$
  - normalized histogram
  - also a probability of occurence



Histogram equalization

- Automatic process of enhancing the contrast of any given image

Histogram Equalization

Next Class

- Chapter 3 G&W second part on "Spatial Filtering"
- Also read chapter 2, section 2.6.5. as introduction