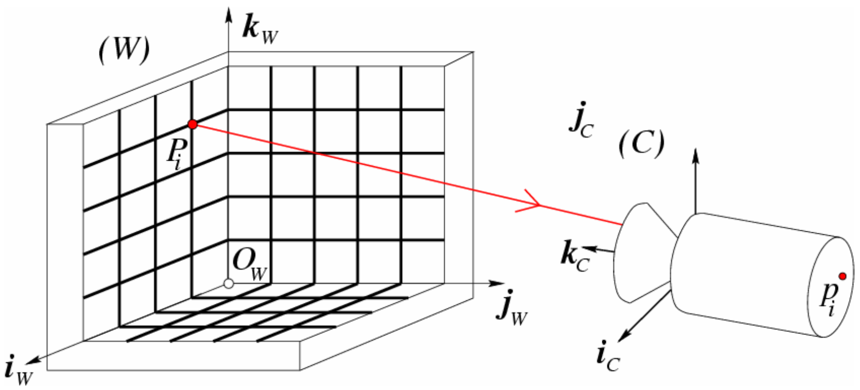

Computer Vision

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Geometric Camera Calibration

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Series outline

- Cameras and lenses
- Geometric camera models
- Geometric camera calibration
- Stereopsis

Lecture outline

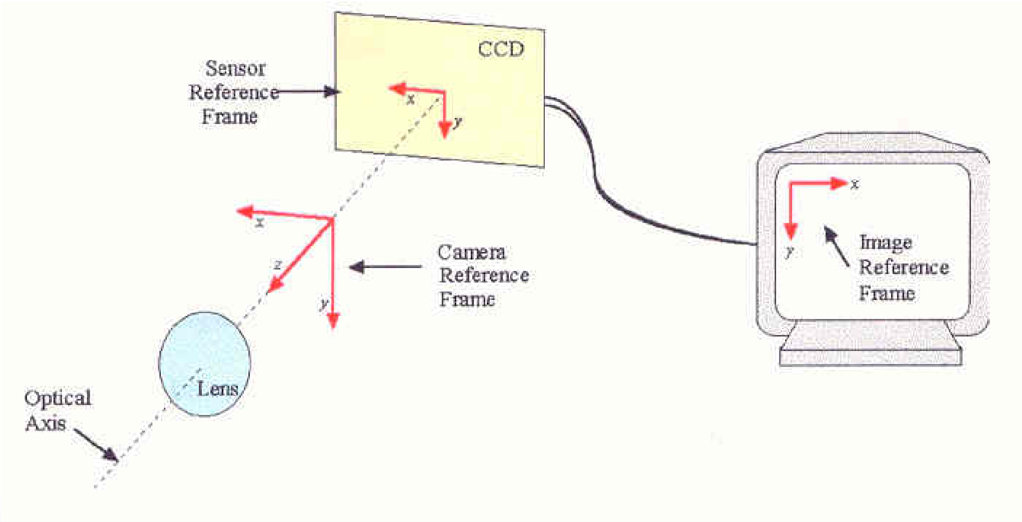
- The calibration problem
- Least-square technique
- Calibration from points
- Radial distortion
- A note on calibration patterns

Camera calibration

Camera calibration is determining the *intrinsic* and *extrinsic* parameters of the camera.

The are three coordinate systems involved: image, camera, and world.

Key idea: to write the *projection equations* linking the known coordinates of a set of 3-D points and their projections, and solve for the camera parameters.



Projection matrix

$$\mathcal{M} = \begin{pmatrix} \alpha \boldsymbol{r}_1^T - \alpha \cot\theta \boldsymbol{r}_2^T + u_0 \boldsymbol{r}_3^T & \alpha t_x - \alpha \cot\theta t_y + u_0 t_z \\ \dfrac{\beta}{\sin\theta} \boldsymbol{r}_2^T + v_0 \boldsymbol{r}_3^T & \dfrac{\beta}{\sin\theta} t_y + v_0 t_z \\ \boldsymbol{r}_3^T & t_z \end{pmatrix}$$

Replacing $\mathcal{M}$ by $\lambda\mathcal{M}$ in

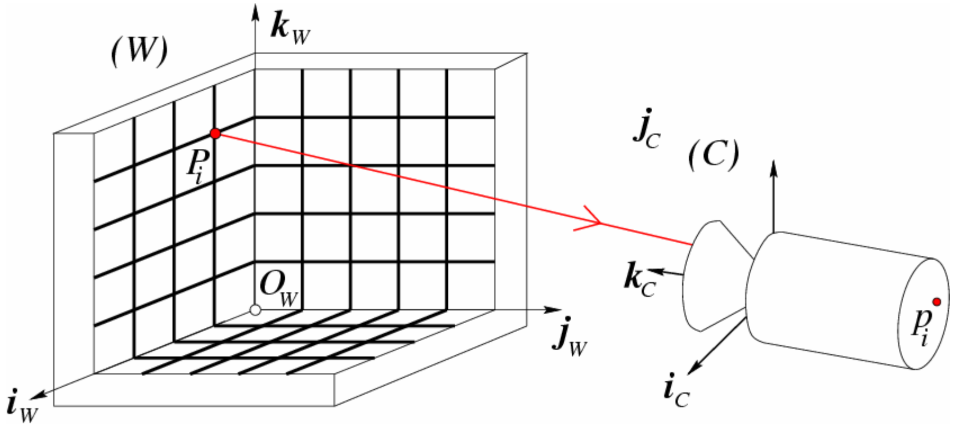
$$\begin{cases} u = \dfrac{\boldsymbol{m}_1 \cdot \boldsymbol{P}}{\boldsymbol{m}_3 \cdot \boldsymbol{P}} \\ v = \dfrac{\boldsymbol{m}_2 \cdot \boldsymbol{P}}{\boldsymbol{m}_3 \cdot \boldsymbol{P}} \end{cases}$$

does not change u and v.

M is only defined up to scale in this setting.

The calibration problem

$$\begin{cases} u_i = \dfrac{\boldsymbol{m}_1(\boldsymbol{i},\boldsymbol{e}) \cdot \boldsymbol{P}_i}{\boldsymbol{m}_3(\boldsymbol{i},\boldsymbol{e}) \cdot \boldsymbol{P}_i} \\ \\ v_i = \dfrac{\boldsymbol{m}_2(\boldsymbol{i},\boldsymbol{e}) \cdot \boldsymbol{P}_i}{\boldsymbol{m}_3(\boldsymbol{i},\boldsymbol{e}) \cdot \boldsymbol{P}_i} \end{cases} \quad \text{for} \quad i = 1, \ldots, n$$

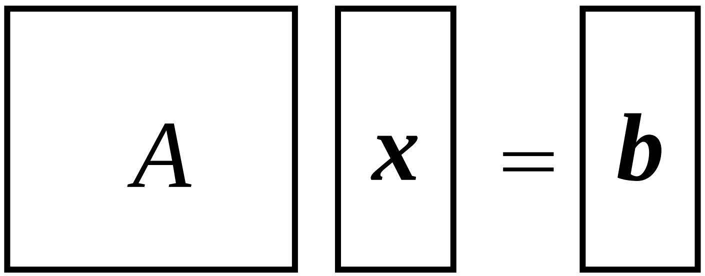


Given n points $P_1, \ldots, P_n$ with *known* positions and their images $p_1, \ldots, p_n$

Find $\boldsymbol{i}$ and $\boldsymbol{e}$ such that

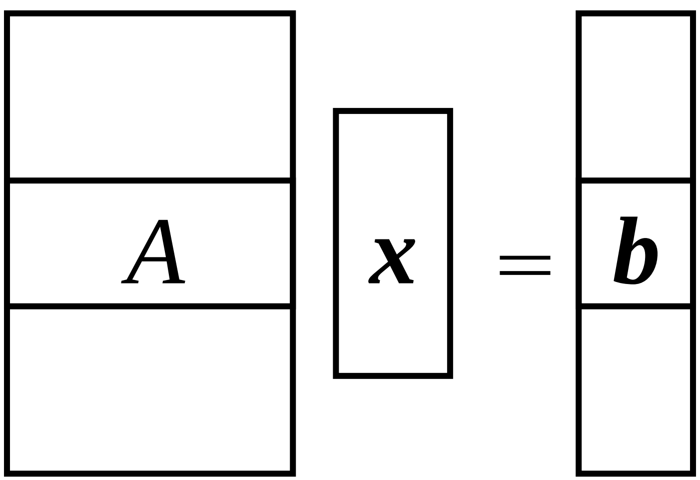
$$\sum_{i=1}^{n} \left[\left(u_i - \frac{\boldsymbol{m}_1(\boldsymbol{i},\boldsymbol{e}) \cdot \boldsymbol{P}_i}{\boldsymbol{m}_3(\boldsymbol{i},\boldsymbol{e}) \cdot \boldsymbol{P}_i} \right)^2 + \left(v_i - \frac{\boldsymbol{m}_2(\boldsymbol{i},\boldsymbol{e}) \cdot \boldsymbol{P}_i}{\boldsymbol{m}_3(\boldsymbol{i},\boldsymbol{e}) \cdot \boldsymbol{P}_i} \right)^2 \right] \quad \text{is minimized}$$

Linear systems



Square system:

- Unique solution
- Gaussian elimination



Rectangular system:

- underconstrained: Infinity of solutions
- Overconstrained: no solution

Minimize $|Ax-b|^2$

How do you solve overconstrained linear equations?

- Define $E = |\boldsymbol{e}|^2 = \boldsymbol{e} \cdot \boldsymbol{e}$ with

$$\begin{aligned} \boldsymbol{e} &= A\boldsymbol{x} - \boldsymbol{b} = \left[\begin{array}{c|c|c|c} \boldsymbol{c}_1 & \boldsymbol{c}_2 & \ldots & \boldsymbol{c}_n \end{array} \right] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \boldsymbol{b} \\ &= x_1\boldsymbol{c}_1 + x_2\boldsymbol{c}_2 + \cdots x_n\boldsymbol{c}_n - \boldsymbol{b} \end{aligned}$$

- At a minimum,

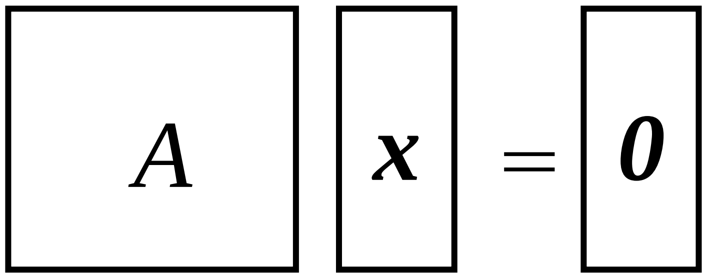
$$\begin{aligned} \frac{\partial E}{\partial x_i} &= \frac{\partial \boldsymbol{e}}{\partial x_i} \cdot \boldsymbol{e} + \boldsymbol{e} \cdot \frac{\partial \boldsymbol{e}}{\partial x_i} = 2\frac{\partial \boldsymbol{e}}{\partial x_i} \cdot \boldsymbol{e} \\ &= 2\frac{\partial}{\partial x_i}(x_1\boldsymbol{c}_1 + \cdots + x_n\boldsymbol{c}_n - \boldsymbol{b}) \cdot \boldsymbol{e} = 2\boldsymbol{c}_i \cdot \boldsymbol{e} \\ &= 2\boldsymbol{c}_i^T(A\boldsymbol{x} - \boldsymbol{b}) = 0 \end{aligned}$$

- or

$$0 = \begin{bmatrix} \boldsymbol{c}_i^T \\ \vdots \\ \boldsymbol{c}_n^T \end{bmatrix} (A\boldsymbol{x} - \boldsymbol{b}) = A^T(A\boldsymbol{x} - \boldsymbol{b}) \Rightarrow A^TA\boldsymbol{x} = A^T\boldsymbol{b},$$

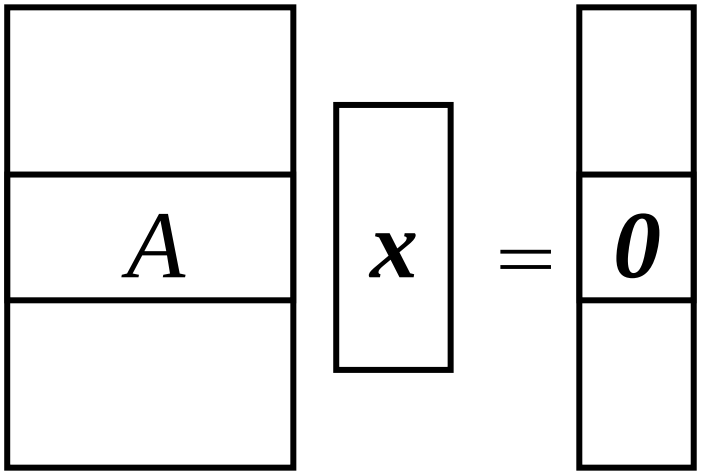
where $\boldsymbol{x} = A^\dagger\boldsymbol{b}$ and $A^\dagger = (A^TA)^{-1}A^T$ is the *pseudoinverse* of A !

Homogeneous linear equations



Square system:

- Unique solution = $\mathbf{0}$
- Unless Det(A) = 0



Rectangular system:

- $\mathbf{0}$ is always a solution

Minimize $|Ax|^2$ under the constraint $|x|^2 = 1$

How do you solve overconstrained homogeneous linear equations?

$$E = |\mathcal{U}\boldsymbol{x}|^2 = \boldsymbol{x}^T(\mathcal{U}^T\mathcal{U})\boldsymbol{x}$$

- Orthonormal basis of eigenvectors: $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_q$.

- Associated eigenvalues: $0 \leq \lambda_1 \leq \ldots \leq \lambda_q$.

- Any vector can be written as

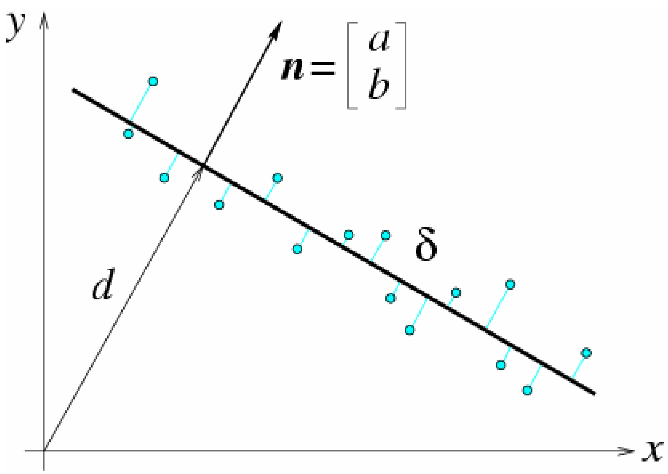
$$\boldsymbol{x} = \mu_1\boldsymbol{e}_1 + \ldots + \mu_q\boldsymbol{e}_q$$

for some μ_i $(i = 1, \ldots, q)$ such that $\mu_1^2 + \ldots + \mu_q^2 = 1$.

$$\begin{aligned} E(\boldsymbol{x}) - E(\boldsymbol{e}_1) &= \boldsymbol{x}^T(\mathcal{U}^T\mathcal{U})\boldsymbol{x} - \boldsymbol{e}_1^T(\mathcal{U}^T\mathcal{U})\boldsymbol{e}_1 \\ &= \lambda_1^2\mu_1^2 + \ldots + \lambda_q^2\mu_q^2 - \lambda_1^2 \\ &\geq \lambda_1^2(\mu_1^2 + \ldots + \mu_q^2 - 1) = 0 \end{aligned}$$

The solution is the eigenvector $\boldsymbol{e}_1$ with least eigenvalue of $U^T U$.

Example: Line fitting



Problem: minimize

$$E(a,b,d) = \sum_{i=1}^{n} (ax_i + by_i - d)^2$$

with respect to (*a*,*b*,*d*).

- Minimize *E* with respect to *d*: $$\frac{\partial E}{\partial d} = 0 \Longrightarrow d = \sum_{i=1}^{n} ax_i + by_i = a\bar{x} + b\bar{y}$$

- Minimize *E* with respect to *a*,*b*: $$E = \sum_{i=1}^{n} [a(x_i - \bar{x}) + b(y_i - \bar{y})]^2 = |\mathcal{U}\boldsymbol{n}|^2$$

where $\mathcal{U} = \begin{pmatrix} x_1 - \bar{x} & y_1 - \bar{y} \\ \ldots & \ldots \\ x_n - \bar{x} & y_n - \bar{y} \end{pmatrix}$ and $\mathcal{U}^T\mathcal{U} = \begin{pmatrix} \sum_{i=1}^{n} x_i^2 - n\bar{x}^2 & \sum_{i=1}^{n} x_i y_i - n\bar{x}\bar{y} \\ \sum_{i=1}^{n} x_i y_i - n\bar{x}\bar{y} & \sum_{i=1}^{n} y_i^2 - n\bar{y}^2 \end{pmatrix}$

Estimation of the projection matrix

Given n points $P_1, \ldots, P_n$ with *known* positions and their images $p_1, \ldots, p_n$

$$\begin{pmatrix} u_i \\ v_i \end{pmatrix} = \begin{pmatrix} \dfrac{\boldsymbol{m}_1 \cdot \boldsymbol{P}_i}{\boldsymbol{m}_3 \cdot \boldsymbol{P}_i} \\ \dfrac{\boldsymbol{m}_2 \cdot \boldsymbol{P}_i}{\boldsymbol{m}_3 \cdot \boldsymbol{P}_i} \end{pmatrix} \iff \begin{pmatrix} \boldsymbol{m}_1 - u_i \boldsymbol{m}_3 \\ \boldsymbol{m}_2 - v_i \boldsymbol{m}_3 \end{pmatrix} \boldsymbol{P}_i = 0$$

The constraints associated with the n points yield a system of $2n$ homogeneous linear equations in the 12 coefficients of the matrix M,

$$\mathcal{P}\boldsymbol{m} = 0 \text{ with } \mathcal{P} \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{P}_1^T & \boldsymbol{0}^T & -u_1 \boldsymbol{P}_1^T \\ \boldsymbol{0}^T & \boldsymbol{P}_1^T & -v_1 \boldsymbol{P}_1^T \\ \ldots & \ldots & \ldots \\ \boldsymbol{P}_n^T & \boldsymbol{0}^T & -u_n \boldsymbol{P}_n^T \\ \boldsymbol{0}^T & \boldsymbol{P}_n^T & -v_n \boldsymbol{P}_n^T \end{pmatrix} \text{ and } \boldsymbol{m} \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{m}_1 \\ \boldsymbol{m}_2 \\ \boldsymbol{m}_3 \end{pmatrix} = 0$$

When $n \geq 6$, homogeneous linear least-square can be used to compute the value of the unit vector $\boldsymbol{m}$ (hence the matrix M) that minimizes $|P\boldsymbol{m}|^2$ as the solution of an eigenvalue problem. The solution is the eigenvector with least eigenvalue of $P^{\mathsf{T}}P$.

Estimation of the intrinsic and extrinsic parameters

Once M is known, you still got to recover the intrinsic and extrinsic parameters!

This is a decomposition problem, NOT an estimation problem.

$$\rho\,\mathcal{M} = \begin{pmatrix} \alpha \boldsymbol{r}_1^T - \alpha \cot\theta \boldsymbol{r}_2^T + u_0 \boldsymbol{r}_3^T & \alpha t_x - \alpha \cot\theta t_y + u_0 t_z \\ \dfrac{\beta}{\sin\theta} \boldsymbol{r}_2^T + v_0 \boldsymbol{r}_3^T & \dfrac{\beta}{\sin\theta} t_y + v_0 t_z \\ \boldsymbol{r}_3^T & t_z \end{pmatrix}$$

- ❑ Intrinsic parameters
- ❑ Extrinsic parameters

Estimation of the intrinsic and extrinsic parameters

Write $M = (A, \boldsymbol{b})$, therefore $\rho(\mathcal{A} \quad \boldsymbol{b}) = \mathcal{K}(\mathcal{R} \quad \boldsymbol{t}) \Longleftrightarrow \rho \begin{pmatrix} \boldsymbol{a}_1^T \\ \boldsymbol{a}_2^T \\ \boldsymbol{a}_3^T \end{pmatrix} = \begin{pmatrix} \alpha \boldsymbol{r}_1^T - \alpha \cot\theta \boldsymbol{r}_2^T + u_0 \boldsymbol{r}_3^T \\ \frac{\beta}{\sin\theta} \boldsymbol{r}_2^T + v_0 \boldsymbol{r}_3^T \\ \boldsymbol{r}_3^T \end{pmatrix}$

Using the fact that the rows of a rotation matrix have unit length and are perpendicular to each other yields

$$\begin{cases} \rho = \varepsilon / |\boldsymbol{a}_3|, \\ \boldsymbol{r}_3 = \rho \boldsymbol{a}_3, \\ u_0 = \rho^2 (\boldsymbol{a}_1 \cdot \boldsymbol{a}_3), \\ v_0 = \rho^2 (\boldsymbol{a}_2 \cdot \boldsymbol{a}_3), \end{cases} \quad \text{where} \quad \varepsilon = \mp 1.$$

Since θ is always in the neighborhood of $\pi/2$ with a positive sine, we have

$$\begin{cases} \rho^2 (\boldsymbol{a}_1 \times \boldsymbol{a}_3) = -\alpha \boldsymbol{r}_2 - \alpha \cot\theta \boldsymbol{r}_1, \\ \rho^2 (\boldsymbol{a}_2 \times \boldsymbol{a}_3) = \dfrac{\beta}{\sin\theta} \boldsymbol{r}_1, \end{cases} \quad \text{and} \quad \begin{cases} \rho^2 |\boldsymbol{a}_1 \times \boldsymbol{a}_3| = \dfrac{|\alpha|}{\sin\theta}, \\ \rho^2 |\boldsymbol{a}_2 \times \boldsymbol{a}_3| = \dfrac{|\beta|}{\sin\theta}. \end{cases}$$

Thus,

$$\begin{cases} \cos\theta = -\dfrac{(\boldsymbol{a}_1 \times \boldsymbol{a}_3) \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3)}{|\boldsymbol{a}_1 \times \boldsymbol{a}_3||\boldsymbol{a}_2 \times \boldsymbol{a}_3|}, \\ \alpha = \rho^2 |\boldsymbol{a}_1 \times \boldsymbol{a}_3| \sin\theta, \\ \beta = \rho^2 |\boldsymbol{a}_2 \times \boldsymbol{a}_3| \sin\theta, \end{cases} \quad \text{and} \quad \begin{cases} \boldsymbol{r}_1 = \dfrac{\rho^2 \sin\theta}{\beta} (\boldsymbol{a}_2 \times \boldsymbol{a}_3) = \dfrac{1}{|\boldsymbol{a}_2 \times \boldsymbol{a}_3|} (\boldsymbol{a}_2 \times \boldsymbol{a}_3), \\ \boldsymbol{r}_2 = \boldsymbol{r}_3 \times \boldsymbol{r}_1. \end{cases}$$

Note that there are two possible choices for the matrix $\mathcal{R}$ depending on the value of ε.

Estimation of the intrinsic and extrinsic parameters

The translation parameters can now be recovered by writing $\mathcal{K}\boldsymbol{t} = \rho \boldsymbol{b}$, and hence $\boldsymbol{t} = \rho \mathcal{K}^{-1} \boldsymbol{b}$. In practical situations, the sign of t_z is often known in advance (this corresponds to knowing whether the origin of the world coordinate system is in front or behind the camera), which allows the choice of a unique solution for the calibration parameters.

Taking radial distortion into account

Assuming that the image centre is known ($u_0 = v_0 = 0$), model the projection process as:

$$p = \frac{1}{z}\begin{pmatrix} 1/\lambda & 0 & 0 \\ 0 & 1/\lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} M\ P$$

where λ is a polynomial function of the squared distance d^2 between the image centre and the image point $\boldsymbol{p}$.

It is sufficient to use low-degree polynomial:

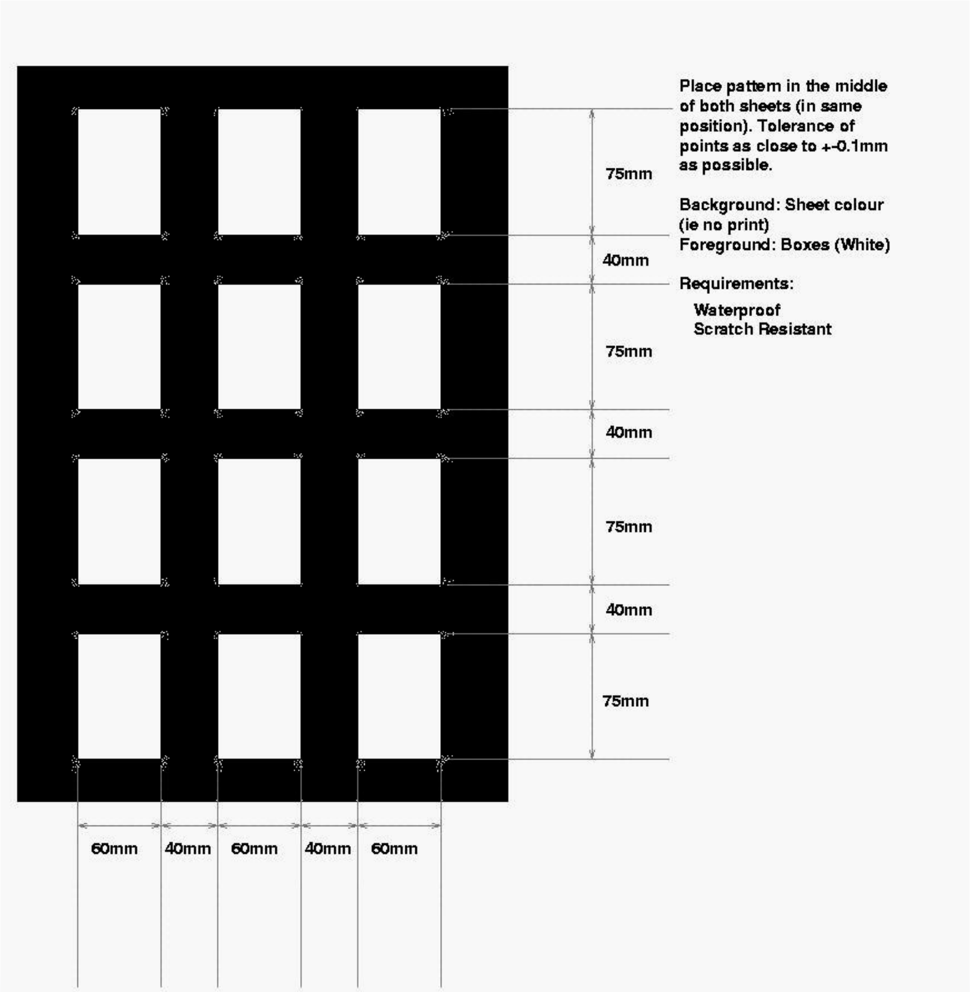
$$\lambda = 1 + \sum_{p=1}^{q} \kappa_p d^{2p}$$, with $q \leq 3$ and the distortion coefficients $\kappa_p\ (p = 1, \ldots, q)$

$$d^2 = \hat{u}^2 + \hat{v}^2$$

$$d^2 = \frac{u^2}{\alpha^2} + \frac{v^2}{\beta^2} + 2\frac{uv}{\alpha\beta}\cos\theta.$$

This yields highly nonlinear constraints on the $q + 11$ camera parameters.

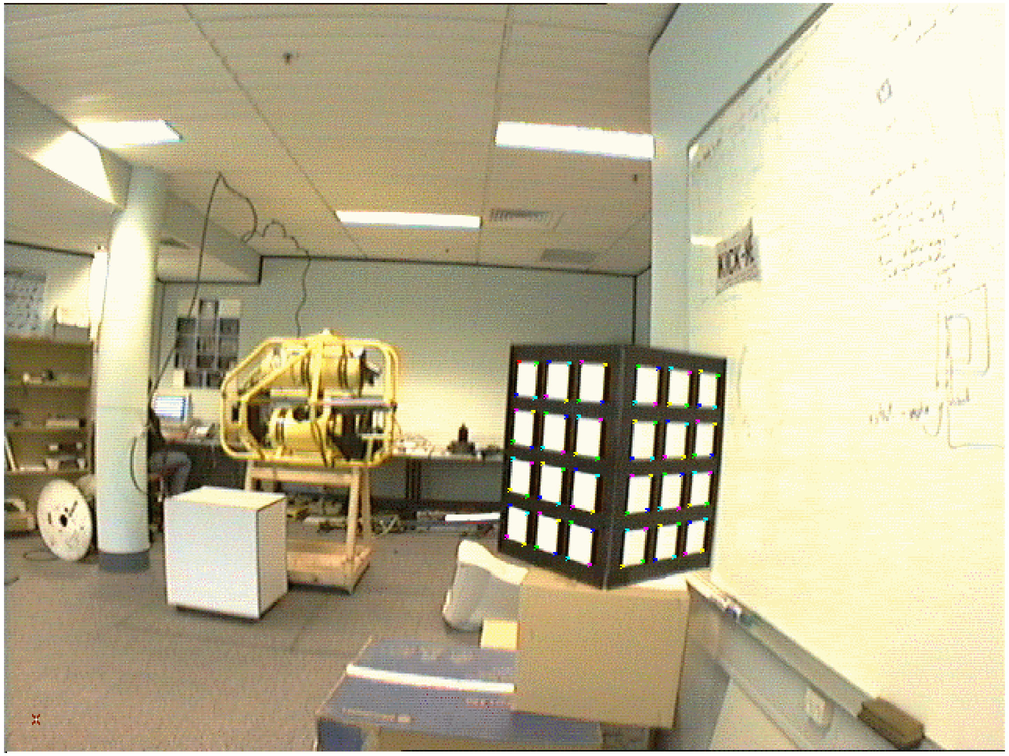
Calibration pattern



The accuracy of the calibration depends on the accuracy of the measurements of the calibration pattern.

Line intersection and point sorting

- Extract and link edges using Canny;
- Fit lines to edges using orthogonal regression;
- Intersect lines.



References

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- “Geometric Frame Work for Vision – Lecture Notes”. A. Zisserman, University of Oxford