matrix. It is of practical interest to put some restrictions on the intrinsic parameters of a camera since, as noted earlier, some of these parameters will be fixed and may be known. In particular, we will say that a 3×4 matrix is a *zero-skew perspective projection matrix* when it can be rewritten (up to scale) as (6.2.7) with $\theta = \pi/2$, and that it is a *perspective projection matrix with zero skew and unit aspect-ratio* when it can be rewritten (up to scale) as (6.2.7) with $\theta = \pi/2$ and $\alpha = \beta$. Of course, a camera with *known* non-zero skew and non-unit aspect-ratio can be transformed into a camera with zero skew and unit aspect-ratio by an appropriate change of image coordinates. Are arbitrary 3×4 matrices perspective projection matrices? The following theorem answers this question.

Theorem 2: *Let* $\mathcal{M} = (\mathcal{A} \quad \boldsymbol{b})$ *be a* 3×4 *matrix and let* $\boldsymbol{a}_i^T$ $(i = 1, 2, 3)$ *denote the rows of the matrix* $\mathcal{A}$ *formed by the three leftmost columns of* $\mathcal{M}$.

- *A necessary and sufficient condition for* $\mathcal{M}$ *to be a perspective projection matrix is that* $\mathrm{Det}(\mathcal{A}) \neq 0$.
- *A necessary and sufficient condition for* $\mathcal{M}$ *to be a zero-skew perspective projection matrix is that* $\mathrm{Det}(\mathcal{A}) \neq 0$ *and*

$$(\boldsymbol{a}_1 \times \boldsymbol{a}_3) \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3) = 0.$$

- *A necessary and sufficient condition for* $\mathcal{M}$ *to be a perspective projection matrix with zero skew and unit aspect-ratio is that* $\mathrm{Det}(\mathcal{A}) \neq 0$ *and*

$$\begin{cases} (\boldsymbol{a}_1 \times \boldsymbol{a}_3) \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3) = 0, \\ (\boldsymbol{a}_1 \times \boldsymbol{a}_3) \cdot (\boldsymbol{a}_1 \times \boldsymbol{a}_3) = (\boldsymbol{a}_2 \times \boldsymbol{a}_3) \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3). \end{cases}$$

The conditions of the theorem are clearly necessary: according to (6.2.6), we have $\mathcal{A} = \mathcal{K}\mathcal{R}$, thus the determinants of $\mathcal{A}$ and $\mathcal{K}$ are the same and $\mathcal{A}$ is non-singular. Further, a simple calculation shows that the rows of $\mathcal{K}\mathcal{R}$ in (6.2.7) satisfy the conditions of the theorem under the various assumptions imposed by its statement. Proofs that they are also sufficient can be found in [Faugeras, 1993; Heyden, 1995] and in the exercises. Note that when the conditions of the theorem are satisfied, there are exactly four sets of intrinsic and extrinsic parameters satisfying (6.2.7), see [Faugeras, 1993; Heyden, 1995] and Section 6.3.1.

6.3 Calibration Methods

This section introduces various techniques for estimating the intrinsic and extrinsic parameters of a camera, a process known as geometric camera calibration. Specifically, suppose that a camera observes n geometric features such as points or lines with known positions in some fixed world coordinate system. This section addresses the problem of (1) computing the perspective projection matrix $\mathcal{M}$ associated with the camera in this coordinate system, then (2) computing the intrinsic and extrinsic

parameters of the camera from this matrix. Once a camera has been calibrated, it is possible to associate with any image point a well-defined ray passing through this point and the camera's optical center, and to conduct quantitative three-dimensional measurements from digitized pictures [Tsai, 1987a].

6.3.1 A Linear Approach to Camera Calibration

Let us first assume that our camera has non-zero skew. According to Theorem 2, the matrix $\mathcal{M}$ is not singular but otherwise arbitrary. If the 4-vectors $\boldsymbol{P}_i$ $(i = 1, \ldots, n)$ and $\boldsymbol{m}_j^T$ $(j = 1, 2, 3)$ denote respectively the homogeneous coordinate vectors of the points P_i and the rows of the matrix $\mathcal{M}$, we can express the position of the image of each point as

$$\begin{cases} u_i = \dfrac{\boldsymbol{m}_1 \cdot \boldsymbol{P}_i}{\boldsymbol{m}_3 \cdot \boldsymbol{P}_i}, \\ v_i = \dfrac{\boldsymbol{m}_2 \cdot \boldsymbol{P}_i}{\boldsymbol{m}_3 \cdot \boldsymbol{P}_i}, \end{cases} \iff \begin{cases} (\boldsymbol{m}_1 - u_i \boldsymbol{m}_3) \cdot \boldsymbol{P}_i = 0, \\ (\boldsymbol{m}_2 - v_i \boldsymbol{m}_3) \cdot \boldsymbol{P}_i = 0. \end{cases}$$

Collecting these constraints for all points yields a system of $2n$ homogeneous linear equations in the twelve coefficients of the matrix $\mathcal{M}$, namely,

$$\mathcal{P}\boldsymbol{m} = 0, \quad \text{where} \quad \mathcal{P} \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{P}_1^T & \boldsymbol{0}^T & -u_1 \boldsymbol{P}_1^T \\ \boldsymbol{0}^T & \boldsymbol{P}_1^T & -v_1 \boldsymbol{P}_1^T \\ \ldots & \ldots & \ldots \\ \boldsymbol{P}_n^T & \boldsymbol{0}^T & -u_n \boldsymbol{P}_n^T \\ \boldsymbol{0}^T & \boldsymbol{P}_n^T & -v_n \boldsymbol{P}_n^T \end{pmatrix} \quad \text{and} \quad \boldsymbol{m} \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{m}_1 \\ \boldsymbol{m}_2 \\ \boldsymbol{m}_3 \end{pmatrix} = 0. \tag{6.3.1}$$

When $n \geq 6$, the system of equations (6.3.1) is in general *overconstrained*, i.e., there is no non-zero vector $\boldsymbol{m} \in \mathbb{R}^{12}$ that satisfies exactly these equations. On the other hand, the zero vector is always a solution. The *linear least-squares* literature, as briefly discussed in the insert next page, provides methods for computing the value of the *unit* vector $\boldsymbol{m}$ that minimizes $|\mathcal{P}\boldsymbol{m}|^2$. In particular, estimating the vector $\boldsymbol{m}$ (hence the matrix $\mathcal{M}$) reduces to computing the eigenvectors and eigenvalues of the 12×12 matrix $\mathcal{P}^T\mathcal{P}$.

Technique: Linear Least Squares Methods

Let us consider a system of n linear equations in p unknowns:

$$\left\{\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1p}x_p = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2p}x_p = b_2 \\ \ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{np}x_p = b_n \end{array}\right. \Leftrightarrow \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1p} \\ a_{21} & a_{22} & \ldots & a_{2p} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{np} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_p \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \ldots \\ b_n \end{pmatrix}. \tag{6.3.2}$$

Let $\mathcal{A}$ denote the $n \times p$ matrix with coefficients a_{ij}, and let $\boldsymbol{x} = (x_1, \ldots, x_p)^T$ and $\boldsymbol{b} = (b_1, \ldots, b_n)^T$. We know from linear algebra that (in general):

1. when $n < p$, there exists an $(p-n)$-dimensional vector space of vectors $\boldsymbol{x}$ that are solutions of (6.3.2);
2. when $n = p$, there is a unique solution;
3. when $n > p$, there is no solution.

This statement is true when the rank of $\mathcal{A}$ is maximal, i.e., equal to $\min(n, p)$ (this is what we mean by "in general"). When the rank is lower, there exists a higher-dimensional set of solutions.

Here we will consider the overconstrained case $n > p$. Since there is no exact solution in this case, we will content ourselves with finding the vector $\boldsymbol{x}$ that minimizes the error measure

$$E \stackrel{\text{def}}{=} \sum_{i=1}^{n} (a_{i1}x_1 + \ldots + a_{ip}x_p - b_i)^2 = |\mathcal{A}\boldsymbol{x} - \boldsymbol{b}|^2.$$

E is proportional to the mean-squared error associated with the equations, hence the name of least-squares methods given to techniques for minimizing E.

Now, we can write $E = |\boldsymbol{e}^T\boldsymbol{e}|$, where $\boldsymbol{e} \stackrel{\text{def}}{=} \mathcal{A}\boldsymbol{x} - \boldsymbol{b}$. To find the vector $\boldsymbol{x}$ minimizing E, we write that the derivatives of this error measure with respect to the coordinates x_i $(i = 1, \ldots, p)$ of $\boldsymbol{x}$ must be zero, i.e.,

$$\frac{\partial E}{\partial x_i} = 2\frac{\partial \boldsymbol{e}}{\partial x_i} \cdot \boldsymbol{e} = 0 \quad \text{for} \quad i = 1, \ldots, p.$$

But if the vectors $\boldsymbol{c}_i$ $(i = 1, \ldots, p)$ denote the columns of $\mathcal{A}$, we have

$$\frac{\partial \boldsymbol{e}}{\partial x_i} = \frac{\partial}{\partial x_i}\left[\begin{pmatrix} \boldsymbol{c}_1 & \ldots & \boldsymbol{c}_p \end{pmatrix} \begin{pmatrix} x_1 \\ \ldots \\ x_p \end{pmatrix} - \boldsymbol{b}\right] = \frac{\partial}{\partial x_i}(x_1\boldsymbol{c}_1 + \ldots + x_p\boldsymbol{c}_p - \boldsymbol{b}) = \boldsymbol{c}_i.$$

In particular, the constraint $\partial E/\partial x_i = 0$ implies that $\boldsymbol{c}_i^T(\mathcal{A}\boldsymbol{x} - \boldsymbol{b}) = 0$, and stacking the constraints associated with the p coordinates of $\boldsymbol{x}$ yields

$$\boldsymbol{0} = \begin{pmatrix} \boldsymbol{c}_1^T \\ \ldots \\ \boldsymbol{c}_p^T \end{pmatrix} (\mathcal{A}\boldsymbol{x} - \boldsymbol{b}) = \mathcal{A}^T(\mathcal{A}\boldsymbol{x} - \boldsymbol{b}) \Longleftrightarrow \mathcal{A}^T\mathcal{A}\boldsymbol{x} = \mathcal{A}^T\boldsymbol{b}.$$

The equations in this linear system are called the normal equations. When $\mathcal{A}$ has maximal rank p, the matrix $\mathcal{A}^T\mathcal{A}$ is easily shown to be invertible, and the solution of the least-squares problem can be written as

$$\boldsymbol{x} = \mathcal{A}^\dagger\boldsymbol{b} \quad \text{where} \quad \mathcal{A}^\dagger \stackrel{\text{def}}{=} [(\mathcal{A}^T\mathcal{A})^{-1}\mathcal{A}^T].$$

The $p \times p$ matrix $\mathcal{A}^\dagger$ is called the pseudoinverse of $\mathcal{A}$. It coincides with $\mathcal{A}^{-1}$ when the matrix $\mathcal{A}$ is square and non-singular. Linear least-squares problems can be solved without explicitly computing the pseudoinverse, using for example QR decomposition or singular value decomposition techniques, which are known to be better behaved numerically.

Let us now consider a slightly different problem, where we have a system of n *homogeneous* linear equations in p unknowns:

$$\left\{ \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1p}x_p = 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2p}x_p = 0 \\ \ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{np}x_p = 0 \end{array} \right. \Leftrightarrow \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1p} \\ a_{21} & a_{22} & \ldots & a_{2p} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{np} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_p \end{pmatrix} = \mathbf{0}. \quad (6.3.3)$$

As before, we denote by $\mathcal{A}$ the $n \times p$ matrix with coefficients a_{ij}, and define $\boldsymbol{x} = (x_1, \ldots, x_p)^T$. When $n = p$ and the matrix $\mathcal{A}$ is non-singular, the system (6.3.3) admits as a unique solution $\boldsymbol{x} = \mathbf{0}$. Conversely, when $n \geq p$, non-trivial (i.e., non-zero) solutions may only exist when $\mathcal{A}$ is singular.

In this context, minimizing the error measure

$$E \stackrel{\text{def}}{=} |\mathcal{A}\boldsymbol{x}|^2 = \sum_{i=1}^{n} [\boldsymbol{a}_i \cdot \boldsymbol{x}]^2$$

only makes sense when some constraint is imposed on the solution $\boldsymbol{x}$ since $\boldsymbol{x} = \mathbf{0}$ yields the zero global minimum of E.

Since, by homogeneity, $E(\lambda\boldsymbol{x}) = \lambda^2 E(\boldsymbol{x})$, it is reasonable to minimize E under the constraint $|\boldsymbol{x}|^2 = 1$, which avoids the trivial solution and forces the uniqueness of the result.

Let us have another look at the error $E = \boldsymbol{x}^T(\mathcal{A}^T\mathcal{A})\boldsymbol{x}$. The $p \times p$ matrix $\mathcal{A}^T\mathcal{A}$ is symmetric positive semidefinite, and it can be diagonalized in an orthonormal basis of eigenvectors $\boldsymbol{e}_i$ $(i = 1, \ldots, p)$ associated with the eigenvalues $0 \leq \lambda_1 \leq \ldots \leq \lambda_p$. Now we can write any unit vector $\boldsymbol{x}$ as $\boldsymbol{x} = \mu_1\boldsymbol{e}_1 + \ldots + \mu_p\boldsymbol{e}_p$ for some μ_i $(i = 1, \ldots, p)$ such that $\mu_1^2 + \ldots + \mu_p^2 = 1$. We have

$$E(\boldsymbol{x}) - E(\boldsymbol{e}_1) = \boldsymbol{x}^T(\mathcal{A}^T\mathcal{A})\boldsymbol{x} - \boldsymbol{e}_1^T(\mathcal{A}^T\mathcal{A})\boldsymbol{e}_1 = \lambda_1\mu_1^2 + \ldots + \lambda_p\mu_p^2 - \lambda_1 \geq \lambda_1(\mu_1^2 + \ldots + \mu_p^2 - 1) = 0.$$

It follows that the unit vector $\boldsymbol{x}$ minimizing the least-squares error E is the eigenvector $\boldsymbol{e}_1$ associated with the minimum eigenvalue of $\mathcal{A}^T\mathcal{A}$ and the corresponding minimum value of E is λ_1.

Various methods are available for computing the eigenvectors and eigenvalues of a symmetric matrix, including Jacobi transformations and reduction to tridiagonal form followed by QR decomposition.

It should finally be noted that least-squares minimization admits a statistical interpretation in terms of maximum likelihood when the coordinates of the data points are modelled as random variables obeying a normal distribution. We will come back to this interpretation in a latter chapter.

In the noise-free case, there will be a unique solution for the matrix $\mathcal{M}$ as long at the rank of the matrix $\mathcal{P}$ is equal to its maximum value of 11 (the matrix $\mathcal{P}$ is singular since by construction $\mathcal{P}\boldsymbol{m} = 0$). A degenerate point configuration will correspond to the case where the matrix has rank 10 or less, or equivalently the nullspace of the matrix has dimension two or greater. Let us consider a vector $\boldsymbol{l}$ in the nullspace and introduce the vectors formed by successive quadruples of its coordinates, i.e., $\boldsymbol{\lambda} = (l_1, l_2, l_3, l_4)^T$, $\boldsymbol{\mu} = (l_5, l_6, l_7, l_8)^T$ and $\boldsymbol{\nu} = (l_9, l_{10}, l_{11}, l_{12})^T$. Since $\boldsymbol{l}$ belongs to the nullspace we have

$$\mathbf{0} = \mathcal{P}\boldsymbol{l} = \begin{pmatrix} \boldsymbol{P}_1^T & \mathbf{0}^T & -u_1\boldsymbol{P}_1^T \\ \mathbf{0}^T & \boldsymbol{P}_1^T & -v_1\boldsymbol{P}_1^T \\ \dots & \dots & \dots \\ \boldsymbol{P}_n^T & \mathbf{0}^T & -u_n\boldsymbol{P}_n^T \\ \mathbf{0}^T & \boldsymbol{P}_n^T & -v_n\boldsymbol{P}_n^T \end{pmatrix} \begin{pmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\mu} \\ \boldsymbol{\nu} \end{pmatrix} = \begin{pmatrix} \boldsymbol{P}_1^T\boldsymbol{\lambda} - u_1\boldsymbol{P}_1^T\boldsymbol{\nu} \\ \boldsymbol{P}_1^T\boldsymbol{\mu} - v_1\boldsymbol{P}_1^T\boldsymbol{\nu} \\ \dots \\ \boldsymbol{P}_n^T\boldsymbol{\lambda} - u_n\boldsymbol{P}_n^T\boldsymbol{\nu} \\ \boldsymbol{P}_n^T\boldsymbol{\mu} - v_n\boldsymbol{P}_n^T\boldsymbol{\nu} \end{pmatrix},$$

or, equivalently, taking into account the values of u_i and v_i yields

$$\begin{cases} \boldsymbol{P}_i^T\boldsymbol{\lambda} - \dfrac{\boldsymbol{m}_1^T\boldsymbol{P}_i}{\boldsymbol{m}_3^T\boldsymbol{P}_i}\boldsymbol{P}_i^T\boldsymbol{\nu} = 0, \\ \boldsymbol{P}_i^T\boldsymbol{\mu} - \dfrac{\boldsymbol{m}_2^T\boldsymbol{P}_i}{\boldsymbol{m}_3^T\boldsymbol{P}_i}\boldsymbol{P}_i^T\boldsymbol{\nu} = 0, \end{cases} \quad \text{for} \quad i = 1, \dots, n.$$

We finally obtain after clearing the denominators and rearranging the terms:

$$\begin{cases} \boldsymbol{P}_i^T(\boldsymbol{m}_3\boldsymbol{\lambda}^T - \boldsymbol{m}_1\boldsymbol{\nu}^T)\boldsymbol{P}_i = 0, \\ \boldsymbol{P}_i^T(\boldsymbol{m}_3\boldsymbol{\mu}^T - \boldsymbol{m}_2\boldsymbol{\nu}^T)\boldsymbol{P}_i = 0, \end{cases} \quad \text{for} \quad i = 1, \dots, n. \tag{6.3.4}$$

As expected, the vector $\boldsymbol{l}$ associated with $\boldsymbol{\lambda} = \boldsymbol{m}_1$, $\boldsymbol{\mu} = \boldsymbol{m}_2$ and $\boldsymbol{\nu} = \boldsymbol{m}_3$ is a solution of these equations. Are there other solutions?

Let us first consider the case where the points P_i $(i = 1, \dots, n)$ all lie in some plane Π, or equivalently, $\boldsymbol{\Pi} \cdot \boldsymbol{P}_i = 0$ for some 4-vector $\boldsymbol{\Pi}$. Clearly, choosing $(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu})$ equal to $(\boldsymbol{\Pi}, \mathbf{0}, \mathbf{0})$, $(\mathbf{0}, \boldsymbol{\Pi}, \mathbf{0})$, or $(\mathbf{0}, \mathbf{0}, \boldsymbol{\Pi})$, or any linear combination of these vectors will yield a solution of (6.3.4). In other words, the nullspace of $\mathcal{P}$ contains the four-dimensional vector space spanned by these vectors and $\boldsymbol{m}$. In practice, this means that coplanar points should not be used in calibration tasks.

In general, for a given non-zero value of the vector $\boldsymbol{l}$, the points P_i that satisfy (6.3.4) must lie on the curve where the two quadric surfaces defined by the corresponding equations intersect. A closer look at (6.3.4) reveals that the straight line where the planes defined by $\boldsymbol{m}_3 \cdot \boldsymbol{P} = 0$ and $\boldsymbol{\nu} \cdot \boldsymbol{P} = 0$ intersect lies on both quadrics. It can be shown that the intersection curve of these two surfaces consists of this line and of a twisted cubic curve Γ passing through the origin [Faugeras, 1993]. A twisted cubic is entirely determined by six points lying on it, and it follows that seven points chosen at random will not fall on Γ. Since, in addition, this curve passes through the origin, choosing $n \geq 6$ random points will in general guarantee that the matrix $\mathcal{P}$ has rank 11 and that the projection matrix can be recovered in a unique fashion.

Once the projection matrix $\mathcal{M}$ has been estimated, (6.2.7) can be used to recover the intrinsic and extrinsic parameters as follows. If we write as before $\mathcal{M} = (\mathcal{A} \quad \boldsymbol{b})$, we have

$$\rho \begin{pmatrix} \boldsymbol{a}_1^T \\ \boldsymbol{a}_2^T \\ \boldsymbol{a}_3^T \end{pmatrix} = \begin{pmatrix} \alpha \boldsymbol{r}_1^T - \alpha \cot\theta \boldsymbol{r}_2^T + u_0 \boldsymbol{r}_3^T \\ \dfrac{\beta}{\sin\theta} \boldsymbol{r}_2^T + v_0 \boldsymbol{r}_3^T \\ \boldsymbol{r}_3^T \end{pmatrix}.$$

In particular, using the fact that the rows of a rotation matrix have unit length and are perpendicular to each other yields immediately

$$\begin{cases} \rho = \varepsilon / |\boldsymbol{a}_3|, \\ \boldsymbol{r}_3 = \rho \boldsymbol{a}_3, \\ u_0 = \rho^2 (\boldsymbol{a}_1 \cdot \boldsymbol{a}_3), \\ v_0 = \rho^2 (\boldsymbol{a}_2 \cdot \boldsymbol{a}_3), \end{cases} \tag{6.3.5}$$

where $\varepsilon = \mp 1$.

In addition, we have

$$\begin{cases} \rho^2 (\boldsymbol{a}_1 \times \boldsymbol{a}_3) = -\alpha \boldsymbol{r}_2 - \alpha \cot\theta \boldsymbol{r}_1, \\ \rho^2 (\boldsymbol{a}_2 \times \boldsymbol{a}_3) = \dfrac{\beta}{\sin\theta} \boldsymbol{r}_1, \end{cases} \quad \text{and} \quad \begin{cases} \rho^2 |\boldsymbol{a}_1 \times \boldsymbol{a}_3| = \dfrac{|\alpha|}{\sin\theta}, \\ \rho^2 |\boldsymbol{a}_2 \times \boldsymbol{a}_3| = \dfrac{|\beta|}{\sin\theta}, \end{cases} \tag{6.3.6}$$

since θ is always in the neighborhood of $\pi/2$ with a positive sine, and it follows that

$$\begin{cases} \cos\theta = -\varepsilon_u \varepsilon_v \dfrac{(\boldsymbol{a}_1 \times \boldsymbol{a}_3) \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3)}{|\boldsymbol{a}_1 \times \boldsymbol{a}_3| |\boldsymbol{a}_2 \times \boldsymbol{a}_3|}, \\ \alpha = \varepsilon_u \rho^2 |\boldsymbol{a}_1 \times \boldsymbol{a}_3| \sin\theta, \\ \beta = \varepsilon_v \rho^2 |\boldsymbol{a}_2 \times \boldsymbol{a}_3| \sin\theta, \end{cases} \tag{6.3.7}$$

where $\varepsilon_u = \alpha/|\alpha|$ and $\varepsilon_v = \beta/|\beta|$.

We can now compute $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ from the second equation in (6.3.6) as

$$\begin{cases} \boldsymbol{r}_1 = \dfrac{\rho^2 \sin\theta}{\beta} (\boldsymbol{a}_2 \times \boldsymbol{a}_3) = \dfrac{1}{|\boldsymbol{a}_2 \times \boldsymbol{a}_3|} (\boldsymbol{a}_2 \times \boldsymbol{a}_3), \\ \boldsymbol{r}_2 = \boldsymbol{r}_3 \times \boldsymbol{r}_1. \end{cases} \tag{6.3.8}$$

Note that there are four possible choices for the matrix $\mathcal{R}$ depending on the values of ε and ε_v.

Finally, the translation parameters are recovered by writing

$$\rho \begin{pmatrix} \alpha t_x - \alpha \cot\theta t_y + u_0 t_z \\ \dfrac{\beta}{\sin\theta} t_y + v_0 t_z \\ t_z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$